Question:

For the non-homogeneous system $Ax=b$, we want to show that $A^TAx = A^Tb$ is always consistent.

proof:

Assume that $A \in$ Matrix $m \times n$, $x \in$ Matrix $n \times 1$,

$\Rightarrow A^T \in$ Matrix $n \times m$, $A^TA \in$ Matrix $n \times n$.

$\Rightarrow Col(A^TA) = \{A^TA \cdot x,$ for all $x \in \mathbb{R}^n\}$.

$Col(A^T) = \{A^T b,$ for all $b \in \mathbb{R}^m\}$.

Hence, we want to show $A^TAx = A^Tb$ is always consistent.

$\Leftrightarrow Col(A^TA) = Col(A^T)$.

$\Leftrightarrow \underbrace{Col(A^TA) \subseteq Col(A^T)}_{\text{Part I}}$, and $\underbrace{\dim(Col(A^TA)) = \dim(Col(A^T))}_{\text{Part II}}$.

Part I:

for any $x \in \mathbb{R}^n$, $A^TAx = A^T(\underbrace{Ax}_{m\times 1}) \in Col(A^T)$,

$\Rightarrow Col(A^TA) \subseteq Col(A^T)$.

Part II : Claim 1 : $Null(A) = Null(A^TA)$

proof of Claim 1:

① pick $x \in Null(A)$,

$\Rightarrow Ax = 0$

$\Rightarrow A^T \cdot Ax = A^T \cdot 0 = 0$

$\Rightarrow x \in Null(A^TA)$

$\Rightarrow Null(A) \subseteq Null(A^TA)$

② Pick $y \in Null(A^TA)$,

$\Rightarrow A^TAy = 0$

$\Rightarrow y^TA^TAy = y^T 0 = 0$

$\Rightarrow (Ay)^T \cdot Ay = 0$

$\Rightarrow Ay = 0$

$\Rightarrow y \in Null(A)$

$\Rightarrow Null(A^TA) \subseteq Null(A)$

So, $Null(A) = Null(A^TA)$.

Hence, $nullity(A) = nullity(A^TA)$.

Since $nullity(A) + dim(col(A)) = n$,

$\Rightarrow dim(col(A)) = n - nullity(A)$.

and $nullity(A^TA) + dim(col(A^TA)) = n$,

$\Rightarrow dim(col(A^TA)) = n - nullity(A^TA)$.

$\Rightarrow dim(col(A)) = dim(col(A^TA))$

Lastly, $dim(col(A)) = dim(row(A^T)) = dim(col(A^T))$

Hence, $dim(col(A^T)) = dim(col(A^TA))$.