Problem:

If we assume $A_{2\times2}$ is a real matrix, and v_1 is a complex eigenvector of A with complex eigenvalue λ_1. $v_2 = \overline{v_1}$, $\lambda_2 = \overline{\lambda_1}$, where $\overline{v_1}$ and $\overline{\lambda_1}$ are the corresponding complex conjugate of v_1 and λ_1 respectively.

Show that $Av_2 = \lambda_2 \cdot v_2$. (*)

Method 1:

We assume that $\lambda_1 = a + bi$, $v_1 = p + q\cdot i$, where $a, b \in \mathbb{R}$, $p, q \in \mathbb{R}^2$.

Since v_1 is an eigenvector of A with eigenvalue λ_1,

$$Av_1 = \lambda_1 v_1$$

$$\Rightarrow A(p+qi) = (a+bi)(p+qi)$$

$$\Rightarrow Ap + Aqi = (ap - bq) + (aq + bp)i$$

$$\Rightarrow \begin{cases} Ap = ap - bq \\ Aq = aq + bp \end{cases}$$

$$\Rightarrow \text{LHS of } (*) = A(p - qi) = Ap - Aq\cdot i$$

$$= ap - bq - (aq + bp)\cdot i$$

$$= ap - bq - aqi - bpi$$

$$\text{RHS of } (*) = (a - bi)(p - qi)$$

$$= ap - bq - aqi - bpi$$

$$= \text{LHS of } (*)$$

$$\Longrightarrow Av_2 = \lambda_2 v_2$$

Method 2:

$A_{2\times2}$ is a real matrix $\Longrightarrow \overline{A} = A$.

Since v_1 is an eigenvector of A with eigenvalue λ_1,

$$Av_1 = \lambda_1 v_1$$

Taking complex conjugate on both sides to get

$$\Rightarrow \overline{Av_1} = \overline{\lambda_1 v_1}$$

$\Rightarrow \overline{A}\,\overline{\vec{v}_1} = \overline{\lambda_1} \cdot \overline{v_1}$ (as the complex conjugate of product = the product of complex conjugate.)

$\Rightarrow A \cdot v_2 = \lambda_2 \cdot v_2 .$