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Determinants and Elementary Row Operation (E.R.O)

Here we will show you the simple proof of the effect of elementary row operations on determinants.

① E.R.O: $A \xrightarrow{kR_j} B$. Determinant: $\det(B) = k\cdot\det(A)$

Proof:

we assume
$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\begin{matrix} \\ \\ j_{th}\text{ row} \\ \\ \\ \end{matrix}, \quad \text{so } B = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ k\cdot a_{j1} & \cdots & k\cdot a_{jn} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\begin{matrix} \\ \\ j_{th}\text{ row} \\ \\ \\ \end{matrix}.$$

For $\det(B)$, expand B along the j_{th} row to get,

$$\det(B) = (-1)^{j+1}\cdot k a_{j1}\cdot \underbrace{\begin{vmatrix} a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ ka_{j2} & \cdots & ka_{jn} \\ \vdots & & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix}}_{\#\text{ denoted by } |\ \ |_{j,1}} + (-1)^{j+2}\cdot k a_{j2}\cdot \left|\ \ \right|_{j,2} + \cdots$$

$$+ (-1)^{j+n}\cdot k a_{jn}\cdot \left|\ \ \right|_{j,n},$$ where $|\ \ |_{j,n}$ denotes the determinant of the matrix after deleting the j_{th} row and the n_{th} column of B.

Then,

$$\det(B) = k\cdot \underline{\left((-1)^{j+1}a_{j1}\cdot\left|\ \ \right|_{j,1} + (-1)^{j+2}a_{j2}\cdot\left|\ \ \right|_{j,2} + \cdots + (-1)^{j+n}a_{jn}\cdot\left|\ \ \right|_{j,n}\right)}$$

$$= k\cdot \underline{\det(A)}.$$

Hence, $\det(B) = k\cdot\det(A)$.

② E.R.O: $A \xrightarrow{R_i + kR_j} B$. Determinant: $\det(B) = \det(A)$

Proof:

$$\det(B) = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1}+ka_{j1} & \cdots & a_{in}+k\cdot a_{jn} \\ \vdots & & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = (a_{i1}+ka_{j1})\cdot(-1)^{i+1}\cdot\left|\ \ \right|_{i,1} + (a_{i2}+ka_{j2})\cdot(-1)^{i+2}\cdot\left|\ \ \right|_{i,2}$$

$$+ \cdots + (a_{in}+ka_{jn})\cdot(-1)^{i+n}\cdot\left|\ \ \right|_{i,n}.$$

Then,

$$\det(B) = \left(a_{i1}\cdot(-1)^{i+1}\left|\;\right|_{i,1} + a_{i2}\cdot(-1)^{i+2}\left|\;\right|_{i,2} + \cdots + a_{in}\cdot(-1)^{i+n}\left|\;\right|_{i,n}\right) +$$

$$\left(k\cdot a_{j1}\cdot(-1)^{i+1}\left|\;\right|_{i,1} + k\cdot a_{j2}\,(-1)^{i+2}\left|\;\right|_{i,2} + \cdots + k\cdot a_{jn}\cdot(-1)^{i+n}\left|\;\right|_{i,n}\right)$$

$$= \det(A) + k\cdot\left(a_{j1}\cdot(-1)^{i+1}\left|\;\right|_{i,1} + a_{j2}\,(-1)^{i+2}\left|\;\right|_{i,2} + \cdots + a_{jn}\cdot(-1)^{i+n}\left|\;\right|_{i,n}\right)$$

$$= \det(A) + k\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & & \vdots \\ a_{j1} & \cdots & a_{jn} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

$$= \det(A) + k\cdot 0$$

$$= \det(A).$$

Hence, $\det(B) = \det(A)$.

③ E.R.O: $A \xrightarrow{R_i \longleftrightarrow R_j} B$. Determinant: $\det(B) = -\det(A)$.

proof:

$$\det(B) = \begin{vmatrix} \vdots \\ R_j \\ \vdots \\ R_i \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ R_j - R_i \\ \vdots \\ R_i \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ R_j - R_i \\ \vdots \\ R_j \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ -R_i \\ \vdots \\ R_j \\ \vdots \end{vmatrix} = -\begin{vmatrix} \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \end{vmatrix} = -\det(A).$$

$R_j + (-1)\cdot R_i$ $\quad$ $R_i + (R_j - R_i)$ $\quad$ $(R_j - R_i) + (-1)\cdot R_j$