Determinant and Matrix Operations (1)

Here we will prove the effect of some matrix operations on the determinants.

Let A and B be $n\times n$ square matrix, and c be a scalar.

① $\det(cA) = c^n \cdot \det(A)$.

proof: When we compute $\det(cA)$, we may first expand along the 1st row where the common scalar c can be taken out. Similar for the expansion of submatrices. Since $A_{n\times n}$ has n rows, then $\underbrace{c \cdot c \cdot \cdots c}_{n\ \text{times}}$ can be taken out from the induction.

Hence, $\det(cA) = c^n \cdot \det(A)$.

② $\det(A^T) = \det(A)$

proof: If we expand the matrix A along the ith row to determine $\det(A)$,
then expand A^T along the ith column to get $\det(A^T)$.
Since ith row of A = ith column of A^T,
then $\det(A) = \det(A^T)$.

③ $\det(AB) = \det(A) \cdot \det(B)$

proof: As it's a slightly long story, I put it on a separate page. For more details, you may refer to the proof in "determinant and matrix operations (2)".

④ If A is non-singular, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

proof: Since A is invertible, $AA^{-1} = I$

According to the property ③,

$$\det(AA^{-1}) = \det(A) \cdot \det(A^{-1}) = \det(I) = 1$$

$$\Longrightarrow \det(A^{-1}) = \frac{1}{\det(A)}.$$