Determinant and Matrix Operations (2)

③ $\det(AB) = \det(A)\cdot\det(B)$

Proof: first, we need the statement (△):

$\det(EA) = \det(E)\cdot\det(A)$, where E is an elementary row operation.

Case 1: suppose A is invertible, then A can be reduced to the identity matrix I,

i.e. $E_n E_{n-1} \cdots E_1 A = I \Rightarrow A = E_1^{-1} E_2^{-1} \cdots E_n^{-1} I$

$$= E_1^* E_2^* \cdots E_n^*$$

(where $E_i^* = E_i^{-1}$, $i = 1, 2, \cdots, n$, and E_i^* is also an elementary row operation.)

By the above statement (△),

$$\det(A) = \det(E_1^*)\cdot\det(E_2^* \cdots E_n^*)$$
$$= \det(E_1^*)\cdot\det(E_2^*)\cdot\det(E_3^* \cdots E_n^*)$$
$$= \cdots = \det(E_1^*)\cdot\det(E_2^*) \cdots \det(E_n^*)$$

Then $\det(AB) = \det(E_1^* E_2^* \cdots E_n^* B)$

$$= \det(E_1^*)\cdot\det(E_2^* \cdots E_n^* B)$$
$$= \det(E_1^*)\cdot\det(E_2^*)\ \det(E_3^* \cdots E_n^* B)$$
$$= \det(E_1^*)\cdot\det(E_2^*) \cdots \det(E_n^*)\cdot\det(B)$$
$$= \det(A)\cdot\det(B).$$

Case 2: Suppose A is not invertible, and then the reduced row echelon form (let us call it C) contains a zero row.

Thus there should be also a zero row in the matrix CB.

Hence, $\det(CB) = 0$.

Similar to the deduction in Case 1,

$$E_n E_{n-1} \cdots E_1 A = C \Rightarrow A = E_1^* E_2^* \cdots E_n^* C.$$

$LHS = \det(AB) = \det(E_1^* E_2^* \cdots E_n^* C B)$

$$= \det(E_1^*)\cdot\det(E_2^*) \cdots \det(E_n^*)\cdot\det(CB)$$
$$= \det(E_1^*) \cdots \det(E_n^*)\cdot 0$$
$$= 0.$$

$RHS = \det(A)\cdot\det(B) = 0\cdot\det(B) = 0.$

Hence, $\det(AB) = \det(A)\cdot\det(B)$.