Question:

The matrix $A_{3\times3}$ has eigenvalues $\underbrace{1}_{\lambda_1}$, $\underbrace{0}_{\lambda_2}$ and $\underbrace{-1}_{\lambda_3}$ with corresponding eigenvectors $\underbrace{\begin{pmatrix}0\\1\\1\end{pmatrix}}_{u_1}$, $\underbrace{\begin{pmatrix}1\\-1\\1\end{pmatrix}}_{u_2}$ and $\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{u_3}$ respectively.

(1) Determine whether A is singular.

(2). Determine whether A is diagonalizable.

(3) Find the matrix A.

Discussion:

(1) As 0 is an eigenvalue of A, A is singular.

(2) As A has 3 distinct eigenvalues,

then A has 3 linearly independent eigenvectors,

Hence, A is diagonalizable.

$\boxed{\text{Remark:}}$ based on the first two parts above, you may see there's no relation between the singularity and diagonalizability of matrices.

(3). Method 1 definition of the diagonalizable matrix.

Since A is diagonalizable, which means,

$$P^{-1}AP = D \text{ , where } P \text{ is non-singular.}$$

So
$$P(P^{-1}AP)P^{-1} = PDP^{-1} \Rightarrow (PP^{-1})\cdot A\cdot(PP^{-1}) = PDP^{-1}$$
$$\Rightarrow A = PDP^{-1}$$

We can use the three eigenvectors to form the matrix P,

$$P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

and use the corresponding eigenvalues to form the matrix D,

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

So the required matrix A is given by,

$$A = PDP^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Method 2: Using stacking method.

According to the definition of eigenvectors and eigenvalues,

$$Au_1 = \lambda_1 u_1, \quad Au_2 = \lambda_2 u_2, \quad Au_3 = \lambda_3 u_3.$$

$$\Rightarrow A(u_1 \ u_2 \ u_3) = (\lambda_1 u_1 \ \lambda_2 u_2 \ \lambda_3 u_3)$$

Since u_1, u_2 and u_3 are 3 linearly independent vectors in $\mathbb{R}^3$,

$$A = (\lambda_1 u_1 \ \lambda_2 u_2 \ \lambda_3 u_3) \cdot (u_1 \ u_2 \ u_3)^{-1}$$

$$= \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Method 3. Using the standard basis vectors.

From the given information, we get

$$T(u_1) = \lambda_1 u_1 = u_1, \quad T(u_2) = \lambda_2 u_2 = 0_{3\times 1}, \quad T(u_3) = \lambda_3 u_3 = -u_3$$

$$\Rightarrow T(e_1) = T(u_3) = -u_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$T(e_2) = T\left((u_3 - u_2 + u_1)\cdot \tfrac{1}{2}\right) = \tfrac{1}{2}\left(T(u_3) - T(u_2) + T(u_1)\right)$$

$$= \tfrac{1}{2}(-u_3 - 0_{3\times 1} + u_1) = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$T(e_3) = T(u_1 - e_2) = T(u_1) - T(e_2)$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

$$\Rightarrow A = (T(e_1) \ T(e_2) \ T(e_3)) = \begin{pmatrix} -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$