Question:

An ellipse is defined by $r = 2\cos\theta\, i + 8\sin\theta\, j + 9k$. Find all points on the ellipse at which $r(\theta)$ is perpendicular to $r'(\theta)$.

Way 1:

$$\vec{r}(\theta) = \begin{pmatrix} 2\cos\theta \\ 8\sin\theta \\ 9 \end{pmatrix} \Rightarrow \vec{r}'(\theta) = \begin{pmatrix} -2\sin\theta \\ 8\cos\theta \\ 0 \end{pmatrix}$$

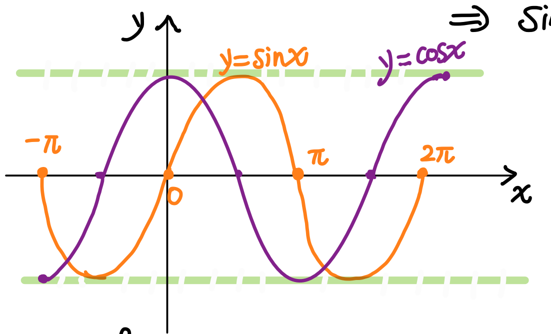
Since $\vec{r}(\theta) \perp \vec{r}'(\theta)$,

$$\Rightarrow \vec{r}(\theta) \cdot \vec{r}'(\theta) = \begin{pmatrix} 2\cos\theta \\ 8\sin\theta \\ 9 \end{pmatrix} \cdot \begin{pmatrix} -2\sin\theta \\ 8\cos\theta \\ 0 \end{pmatrix}$$

$$= -4\sin\theta \cdot \cos\theta + 64\sin\theta \cdot \cos\theta + 0$$

$$= 60 \cdot \sin\theta \cdot \cos\theta = 0$$

$$\Rightarrow \sin\theta \cdot \cos\theta = 0 \Rightarrow \sin\theta = 0 \quad \text{or} \quad \cos\theta = 0$$



$$\Rightarrow \begin{cases} \sin\theta = 0 \\ \cos\theta = \pm 1 \end{cases} \text{or} \begin{cases} \sin\theta = \pm 1 \\ \cos\theta = 0 \end{cases}$$

Recall: $\mathbf{r} = 2\cos\theta\mathbf{i} + 8\sin\theta\mathbf{j} + 9\mathbf{k}$,

and hence,

① When $\sin\theta = 0, \cos\theta = 1$, $r(\theta) = \begin{pmatrix} 2 \\ 0 \\ 9 \end{pmatrix}$; ② When $\sin\theta = 0, \cos\theta = -1$, $r(\theta) = \begin{pmatrix} -2 \\ 0 \\ 9 \end{pmatrix}$;

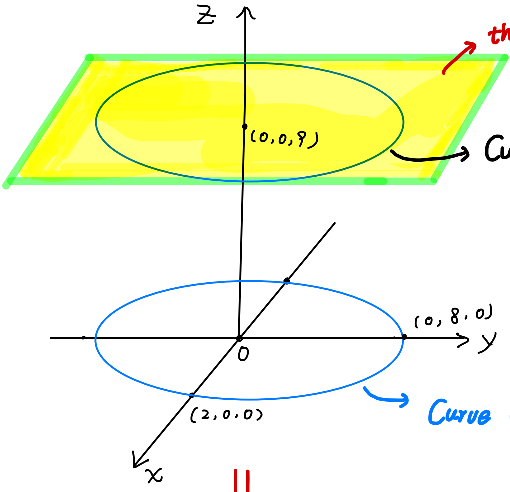
③ When $\cos\theta = 0$, $\sin\theta = 1$, $r(\theta) = \begin{pmatrix} 0 \\ 8 \\ 9 \end{pmatrix}$; ④ When $\cos\theta = 0$, $\sin\theta = -1$, $r(\theta) = \begin{pmatrix} 0 \\ -8 \\ 9 \end{pmatrix}$.

Hence, here are 4 points: $A(2, 0, 9)$, $B(-2, 0, 9)$

$C(0, 8, 9)$, $D(0, -8, 9)$

Way 2 :

$$\gamma(\theta) = \begin{pmatrix} 2\cos\theta \\ 8\sin\theta \\ 9 \end{pmatrix},$$ which is an ellipse on the plane $z=9$.

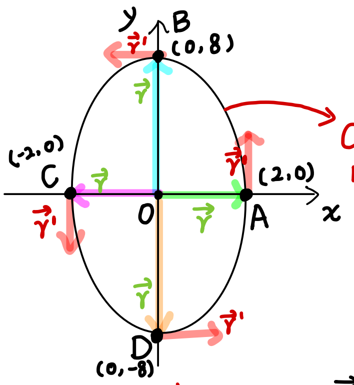


the plane $z=9$

Curve C : $\vec{\gamma}(\theta) = \begin{pmatrix} 2\cos\theta \\ 8\sin\theta \\ 9 \end{pmatrix}$ or $\begin{cases} (\frac{x}{2})^2 + (\frac{y}{8})^2 = 1 \\ z = 9 \end{cases}$

Curve C_1 : $\vec{\gamma}_1(\theta) = \begin{pmatrix} 2\cos\theta \\ 8\sin\theta \\ 0 \end{pmatrix}$ or $\begin{cases} (\frac{x}{2})^2 + (\frac{y}{8})^2 = 1 \\ z = 0 \end{cases}$

$\Downarrow$



Curve C_1 in 2D : $\vec{\gamma}_1(\theta) = \begin{pmatrix} 2\cos\theta \\ 8\sin\theta \end{pmatrix}$ or $(\frac{x}{2})^2 + (\frac{y}{8})^2 = 1$

$\Longrightarrow$ $\vec{\gamma}\cdot\vec{\gamma}' = 0$ on C $\Leftrightarrow$ $\vec{\gamma}_1 \cdot \vec{\gamma}_1' = 0$ on C_1

By observation, $|\vec{\gamma}_1| = |\vec{OP}|$, where p is an arbitrary point on C_1.

and $|\vec{\gamma}_1|$ is locally maximized / minimized at A, B, C, D.

$\Rightarrow$ $|\vec{\gamma}_1|^2 = (\vec{\gamma}_1)^2$ is also locally maximized / minimized at A, B, C, D.

$\Rightarrow$ $\frac{d((\vec{\gamma}_1)^2)}{d\theta} = 2\vec{\gamma}_1\cdot\vec{\gamma}_1' = 0$ at A, B, C, D

$\Rightarrow$ $\vec{\gamma}_1\cdot\vec{\gamma}_1' = 0$ at the points A(2,0), C(-2,0), B(0,8), D(0,-8).

$\Rightarrow$ $\vec{\gamma}\cdot\vec{\gamma}' = 0$ on C at the points (2,0,9), (-2,0,9), (0,8,9), (0,-8,9).