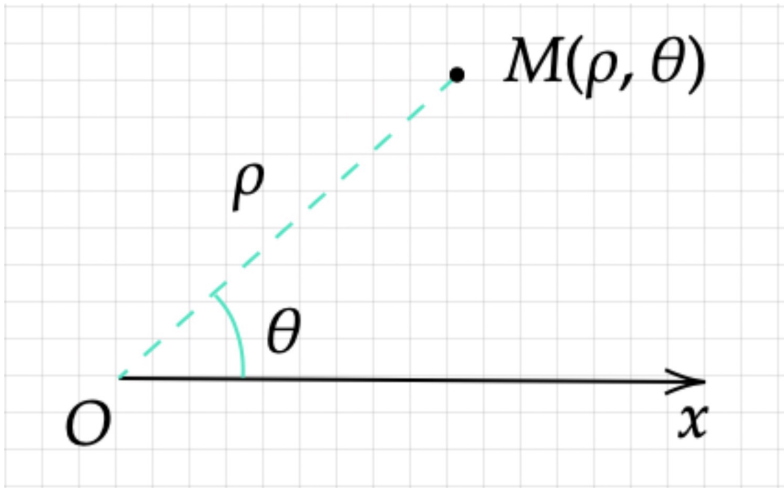
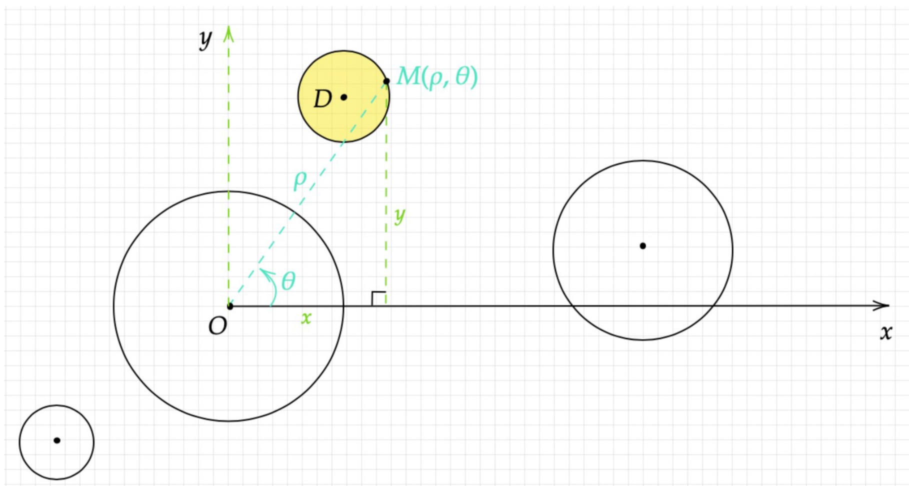
▲ Double integral on the circular region

Firstly, here is the polar coordinate system:

(Note that here we use ρ instead of r to avoid some confusion.)



Then you can put the circle wherever you want.

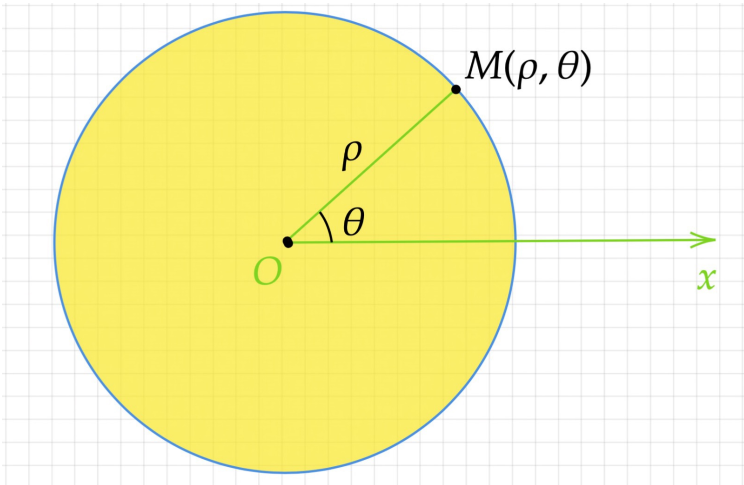


★ No matter where it is, the following equalities alway hold:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$\Rightarrow \iint_D f(x,y)\,dA = \iint_D f(x,y)\,dx\,dy = \iint_D f(\rho\cos\theta, \rho\sin\theta)\,\rho\,d\rho\,d\theta$$

Case 1. the circle is centered at (0, 0), with radius a.

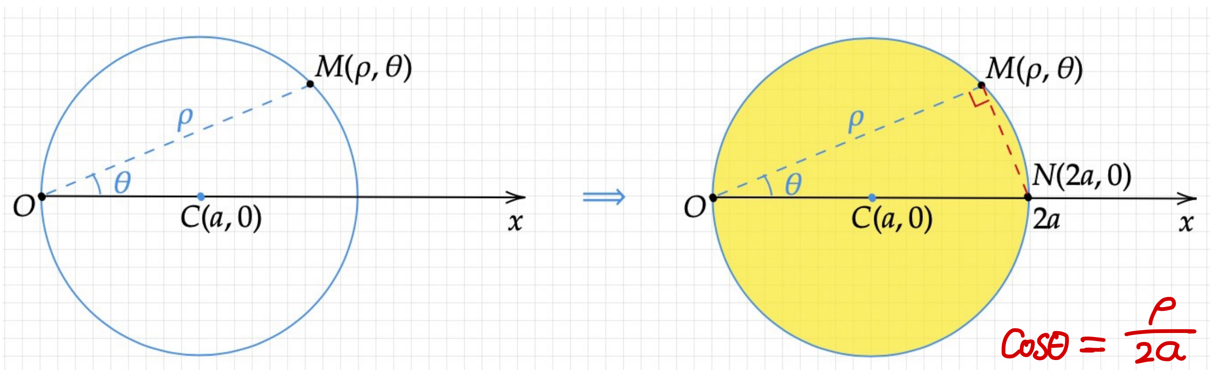


If we take the arbitrary point $M(\rho, \theta)$ on the circle,
then the polar coordinate equation of the circle is $\rho = a$.
And then the domain (highlighted in yellow) is

$$D = \{(\rho, \theta) : 0 \le \rho \le a,\ 0 \le \theta \le 2\pi\}$$

$$\Rightarrow \iint_D f(x, y)\, dA = \int_0^{2\pi} \int_0^{a} f(\rho\cos\theta, \rho\sin\theta)\, \rho\, d\rho\, d\theta$$

Case 2. the circle is centered at $(a, 0)$, with radius a.



$$\cos\theta = \frac{\rho}{2a}$$

$$\Rightarrow \rho = 2a \cdot \cos\theta$$

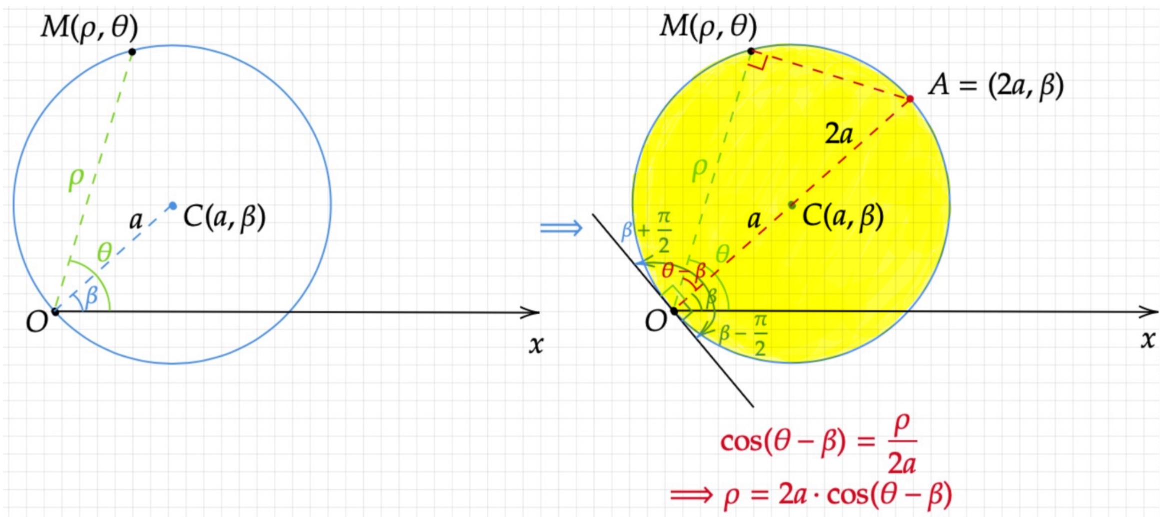
If we take the arbitrary point $M(\rho, \theta)$ on the circle, then the polar coordinate equation of the circle is $\rho = 2a \cdot \cos\theta$.

And then the domain (highlighted in yellow) is

$$D = \{(\rho, \theta) : -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le \rho \le 2a\cos\theta\}$$

$$\Rightarrow \iint_D f(x,y)\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2a\cos\theta} f(\rho\cos\theta, \rho\sin\theta)\,\rho\,d\rho\,d\theta$$

case 3. the circle is centered at (a, β), with radius a.



If we take the arbitrary point $M(\rho, \theta)$ on the circle,

then the polar coordinate equation of the circle is $\rho = 2a \cdot \cos(\theta - \beta)$.

And then the domain (highlighted in yellow) is

$$D = \{(\rho, \theta) : \beta - \frac{\pi}{2} \le \theta \le \beta + \frac{\pi}{2},\ 0 \le \rho \le 2a \cdot \cos(\theta - \beta)\}$$

$$\Rightarrow \iint_D f(x,y)\,dA = \int_{\beta-\frac{\pi}{2}}^{\beta+\frac{\pi}{2}} \int_0^{2a\cdot\cos(\theta-\beta)} f(\rho\cos\theta, \rho\sin\theta)\,\rho\,d\rho\,d\theta$$