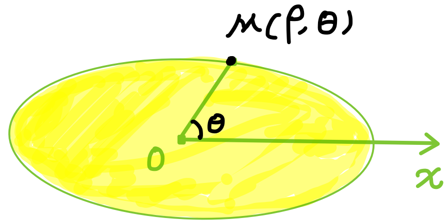
▲ Double integral on the elliptical region:

For example, the equation of the ellipse is give by.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



Substitute $\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases}$ into the above equation,

$$\Rightarrow \frac{\rho^2\cdot\cos^2\theta}{a^2}+\frac{\rho^2\cdot\sin^2\theta}{b^2}=1$$

$$\Rightarrow \rho^2\cdot\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)=1$$

$$\Rightarrow \rho^2=\frac{1}{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}=\frac{a^2b^2}{a^2\sin^2\theta+b^2\cos^2\theta}$$

If we take the arbitrary point $M(\rho,\theta)$ on the ellipse,
then the polar coordinate equation of the ellipse is $\rho=\frac{ab}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$

then the domain (highlighted in yellow) is

$$D=\left\{(\rho,\theta):\ 0\le\theta\le 2\pi,\ 0\le\rho\le \underbrace{\boxed{\frac{ab}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}}}_{=g(\theta)}\right\}$$

$$\Rightarrow \iint_D f(x,y)\,dA=\int_0^{2\pi}\int_0^{g(\theta)} f(\rho\cos\theta,\rho\sin\theta)\,\rho\,d\rho\,d\theta$$

⇒ As you see it will be super complex to do the iterated integral because of $g(\theta)$.

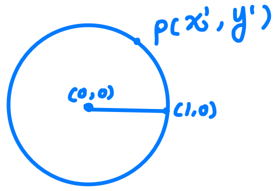
In this case, we need perform some coordinate transformations to convert the domain into a disc, and then do the double integral in polar coordinate.

You may be wondering how? Let's take a deep breath and proceed as bellow:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

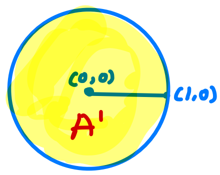
Step 1. change the coordinate

$$\begin{cases} x' = \frac{x}{a} \\ y' = \frac{y}{b} \end{cases} \Rightarrow \begin{cases} x = ax' \\ y = by' \end{cases} \quad \text{and} \quad x'^2 + y'^2 = 1.$$



$$\Rightarrow dA = dx\,dy = d(ax')\,d(by')$$
$$= ab\,dx'\,dy'$$
$$= ab\,dA'$$

Step 2.



$$\iint_A f(x,y)\,dA = \iint_{A'} f(ax', by')\cdot ab\,dA'$$

$$= ab\iint_{A'} f(ax', by')\,dA' \qquad \Rightarrow A' = \{(x', y'):\ x'^2 + y'^2 \leq 1\}$$

$$= ab\iint_{A'} (a\rho\cos\theta,\ b\rho\sin\theta)\,\rho\,d\rho\,d\theta \qquad \Rightarrow A' = \{(\rho,\theta):\ 0 \leq \theta \leq 2\pi,\ 0 \leq \rho \leq 1\}$$

$$= ab\int_0^{2\pi}\int_0^1 (a\rho\cos\theta,\ b\rho\sin\theta)\,\rho\,d\rho\,d\theta$$