Question:

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that the standard matrix for T is $A^* = \begin{pmatrix} 1 & 1 \\ 0.25 & 1 \end{pmatrix}$.

(a) please show that how the input vector is being transformed to the output vector under the given linear transform.

(b). Using the result obtained in (a), sketch the images of the rectangle with $A(4,2)$, $B(-4,2)$, $C(-4,-2)$, $D(4,-2)$.

Discussion:

(a). Firstly, $\det(\lambda I - A^*) = \begin{vmatrix} \lambda-1 & -1 \\ -0.25 & \lambda-1 \end{vmatrix} = (\lambda-1)^2 - \frac{1}{4} = 0 \Rightarrow \lambda_1 = \frac{3}{2}, \ \lambda_2 = \frac{1}{2}$.

Next, for $\lambda_1 = \frac{3}{2}$, Solve the homogeneous system $(\frac{3}{2}I - A^*)x = 0$ to get the general solution $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2t \\ t \end{pmatrix} = t \cdot \underbrace{\boxed{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}}_{u_1} = t \cdot u_1$, $t \in \mathbb{R}$

for $\lambda_2 = \frac{1}{2}$, solve the homogeneous system $(\frac{1}{2}I - A^*)x = 0$ to get the general solution $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2s \\ s \end{pmatrix} = s \cdot \underbrace{\boxed{\begin{pmatrix} -2 \\ 1 \end{pmatrix}}}_{u_2} = s \cdot u_2$, $s \in \mathbb{R}$.

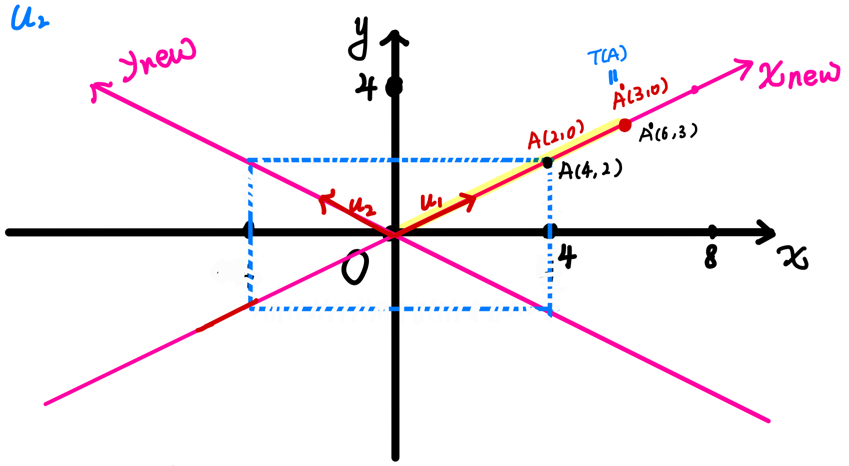
Since A^* is 2×2, and A has 2 linearly independent eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, then A^* is diagonalizable.

Let $P = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, and then $P^{-1}AP = D$.

Then $A^* = PDP^{-1} = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}^{-1}$.

Firstly, we draw the standard coordinate system (black arrows) corresponding to the standard basis vectors.

Then we construct a new coordinate system using the two eigenvectors $\underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{u_1}$ and $\underbrace{\begin{pmatrix} -2 \\ 1 \end{pmatrix}}_{u_2}$ as a basis (red arrows), as shown in the diagram:



Let's take the coordinate vector of point A(4,2) as an example,

i.e $a = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, and then $Aa = PDP^{-1}a$, $D = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $P = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}$.

Step 1. $a \longrightarrow P^{-1}a$, the resulting vector gives the coordinate vector of A relative to the new system.

$$P^{-1}a = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Hence, the coordinate vector of A relative to the new system is $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Step 2. $\underbrace{P^{-1}a}_{b} \rightarrow DP^{-1}a$. The effect is to do a scaling in the new coordinate system.

We know $D = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, $\underbrace{P^{-1}a}_{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, so D scales along u_1 by a factor $\frac{3}{2}$, and scales along u_2 by a factor $\frac{1}{2}$.

Hence, after scaling, the coordinate vector of A' is $\begin{pmatrix} 2\times\frac{3}{2} \\ 0\times\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

Meanwhile, you may perform the matrix multiplication to check that:

$$DP^{-1}a = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

Step 3. $DP^{-1}a \longrightarrow P\,DP^{-1}a$ the resulting vector gives the coordinate vector of A' relative to the original system.

$$P\,DP^{-1}a = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

Hence, the coordinate vector of A' relative to the original system is $\begin{pmatrix} 6 \\ 3 \end{pmatrix}$.

In a nutshell,

T is a scaling along the axes in the direction of $\underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{u_1}$ and $\underbrace{\begin{pmatrix} -2 \\ 1 \end{pmatrix}}_{u_2}$ by the factors $\frac{3}{2}$ and $\frac{1}{2}$ respectively.

(b). Use the conclusion in part (a) to draw the new vertices directly

For example, T scales A along x_{new} by $\frac{3}{2}$ and along y_{new} by $\frac{1}{2}$.
Since $A(4,2)$ lies on the axis x_{new},
we only need to use a ruler to scale OA along x_{new} by $\frac{3}{2}$ to attain OA'.
Similarly, we can sketch the rest 3 vertices B', C', D' to attain the image of rectangle under the linear transformation T.