Question

The temperature at a point (x, y) on a metallic floor is $T(x, y) = 4x^2 - 4xy + y^2$. Wearing thermocouple shoes that measures temperature, you walk along a circle of radius 5 centered at the origin on the metallic floor. What are the highest and lowest temperatures measured by your shoes?

Solution :

Lagrange Multiplier

The target: $T(x,y) = 4x^2 - 4xy + y^2 \Rightarrow T_x = 8x - 4y$, $T_y = -4x + 2y$

The constraint: $g(x,y) = x^2 + y^2 - 25 = 0 \Rightarrow g_x = 2x$, $g_y = 2y$

By the method of Lagrange Multiplier,

$$(*) \begin{cases} 8x - 4y = \lambda \cdot 2x \\ -4x + 2y = \lambda \cdot 2y \\ x^2 + y^2 - 25 = 0 \end{cases} \Rightarrow \begin{cases} 4x - 2y = \lambda \cdot x & (1) \\ 4x - 2y = \lambda \cdot (-2y) & (2) \\ x^2 + y^2 = 25 & (3) \end{cases}$$

$(1),(2) \Rightarrow \lambda \cdot x = \lambda \cdot (-2y)$

$\Rightarrow \lambda \cdot (x + 2y) = 0$

$\Rightarrow \lambda = 0$ or $x = -2y$

Sub $\lambda = 0$ into (1), $4x - 2y = 0 \Rightarrow y = 2x$

Putting them together, $x = -2y$ or $y = 2x$

Method 1.

Case 1. Sub $x = -2y$ into (3) $\Rightarrow (-2y)^2 + y^2 = 25 \Rightarrow 5y^2 = 25 \Rightarrow y^2 = 5$

$$\Rightarrow \begin{cases} x = -2\sqrt{5} \\ y = \sqrt{5} \end{cases} \text{ or } \begin{cases} x = 2\sqrt{5} \\ y = -\sqrt{5} \end{cases}$$

Case 2. Sub $\boxed{y = 2x}$ into (3) $\Rightarrow x^2 + (2x)^2 = 25 \Rightarrow 5x^2 = 25 \Rightarrow x^2 = 5$

$$\Rightarrow \begin{cases} x = \sqrt{5} \\ y = 2\sqrt{5} \end{cases} \text{ or } \begin{cases} x = -\sqrt{5} \\ y = -2\sqrt{5} \end{cases}$$

(1) Sub $x = -2\sqrt{5}$, $y = \sqrt{5}$ into $T(x,y)$,

$T(x,y) = 4x^2 - 4xy + y^2 = (2x - y)^2 = (-4\sqrt{5} - \sqrt{5})^2 = (-5\sqrt{5})^2 = 25 \times 5 = 125$

(2) Sub $x = 2\sqrt{5}$, $y = -\sqrt{5}$ into $T(x,y)$,

$T(x,y) = (2x - y)^2 = (4\sqrt{5} + \sqrt{5})^2 = (5\sqrt{5})^2 = 125$

(3) Sub $x = \sqrt{5}$, $y = 2\sqrt{5}$ into $T(x,y)$,

$T(x,y) = (2x - y)^2 = 0$

(4) Sub $x = \sqrt{5}$, $y = -2\sqrt{5}$ into $T(x,y)$

$T(x,y) = (2x - y)^2 = 0$

Hence, the highest temperature is 125, and the lowest one is 0.

Method 2. Shortcut

Recall:

from the equation ① and ②, we get $x = -2y$ or $y = 2x$.

Case 1. Sub $x = -2y$ into ③, $(-2y)^2 + y^2 = 25 \Rightarrow y^2 = 5$

$$T(x,y) = 4x^2 - 4xy + y^2$$
$$= (2x - y)^2$$
$$= (2\cdot(-2y) - y)^2$$
$$= 25y^2 = 25 \times 5 = 125.$$

Case 2. Sub $y = 2x$ into ③, $x^2 + (2x)^2 = 25 \Rightarrow x^2 = 5$

$$T(x,y) = 4x^2 - 4xy + y^2$$
$$= (2x - y)^2$$
$$= (y - y)^2$$
$$= 0.$$

Hence, the highest temperature is 125,

and the lowest one is 0.

★ Remark:

Result 1.8A

The maximum/minimum value of $f(x,y)$ subject to the constraint $g(x,y) = 0$ occurs at a point (x, y) that satisfies the following three equations

$$f_x = \lambda g_x$$
$$f_y = \lambda g_y$$
$$g(x,y) = 0$$

for some constant λ, called a **Lagrange multiplier**.

② Parametric Equation.

The target: $T(x,y) = 4x^2 - 4xy + y^2$ ①

The constraint: $g(x,y) = x^2 + y^2 - 25 = 0$ ②

$$\Downarrow$$

The parametric equation for ② is $\begin{cases} x = 5\cos\theta \\ y = 5\sin\theta \end{cases}$,

and then sub into ①,

$$\begin{aligned} T(x,y) &= (2x-y)^2 = (10\cos\theta - 5\sin\theta)^2 \\ &= (5\cdot(2\cos\theta - \sin\theta))^2 \\ &= 25\cdot(2\cos\theta - \sin\theta)^2 \\ &= 25\cdot(\sqrt{5}\cdot\cos(\theta+\alpha))^2 \\ &= 25\cdot 5\cdot\cos^2(\theta+\alpha) \\ &= 125\cdot\cos^2(\theta+\alpha) \end{aligned}$$

$= \sqrt{5}\cdot\cos(\theta+\alpha)$, $\alpha = \tan^{-1}(\frac{1}{2})$.

$$\boxed{\begin{array}{l} A\cos\theta - B\sin\theta = \sqrt{A^2+B^2}\cdot\cos(\theta+\alpha) \\ \text{where } \alpha = \tan^{-1}(\frac{B}{A}) \end{array}}$$

$-1 \le \cos(\theta+\alpha) \le 1 \Rightarrow 0 \le \cos^2(\theta+\alpha) \le 1 \Rightarrow 0 \le T(x,y) \le 125$

Hence, the highest temperature is 125,

and the lowest one is 0.

③ Geometric Intepretation

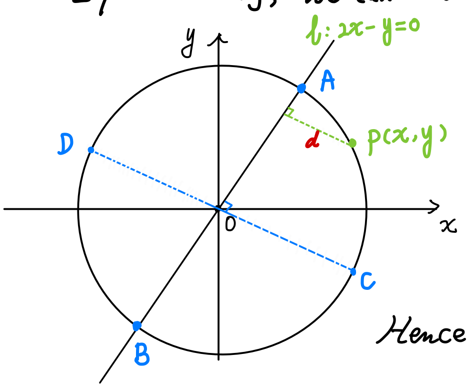
$T(x,y) = (2x-y)^2$ is maximized / minimized

$\Longleftrightarrow$ $|2x-y|$ is maximized / minimized

$\Longleftrightarrow$ $\frac{|2x-y|}{\sqrt{5}} = d$ is maximized / minimized

, which is the distance d between the point $P(x,y)$ and the line l $2x-y=0$.

Equivalently, we can investigate the above distance when P satisfies $x^2+y^2-25=0$



By inspection, d_{max} = radius = 5 (at C, D)

$d_{min} = 0$ (at A, B)

$\Rightarrow T_{max} = (5\times\sqrt{5})^2 = 125$

$T_{min} = (0\times\sqrt{5})^2 = 0$

Hence, the highest temperature is 125, and the lowest one is 0.