Question:

Find the exact value of the line integral

$$\oint_C (xy - \tan(y^2))\,dx + (x^2 - 2xy\cdot\sec^2(y^2))\,dy \quad (\Delta)$$

where C is the counterclockwise oriented triangle with vertices at $(0,0)$, $(1,0)$ and $(0,2)$.

Solution:

Method 1:

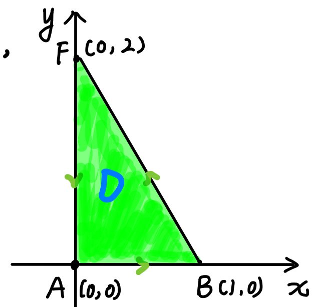
Assume $(\Delta) = \oint_C P\,dx + Q\,dy$, and then

$$P_y = x - \sec^2(y^2)\cdot 2y = x - 2y\cdot\sec^2(y^2),$$

$$Q_x = 2x - 2y\cdot\sec^2(y^2).$$

Hence, $Q_x - P_y = x$

By Green's Theorem,



$$
\begin{aligned}
(\Delta) &= \iint_D x\,dA \\
&= \int_0^1 \underbrace{\int_0^{-2x+2} x\,dy}_{(*)}\,dx \\
&= \int_0^1 (-2x^2+2x)\,dx \\
&= -\frac{2}{3}x^3 + x^2 \Big|_{x=0}^{x=1} \\
&= \frac{1}{3}
\end{aligned}
$$

Region $D = \{(x,y): 0 \le x \le 1,\ 0 \le y \le -2x+2\}$

$$
\begin{aligned}
(*) &= x\cdot\int_0^{-2x+2} dy \\
&= x\cdot(-2x+2) \\
&= -2x^2+2x
\end{aligned}
$$

Method 2. Alternatively, assume

$(\Delta) = \oint_C P\,dx + Q\,dy$, and then

$$P_y = x - 2y\cdot\sec^2(y^2),$$

$$Q_x = 2x - 2y\cdot\sec^2(y^2).$$

By inspection, if we want the final $P_y = Q_x$, we can add x to the original P_y (or add $-x$ to the original Q_x).

$\Rightarrow$ add xy to P (or add $-\frac{x^2}{2}$ to Q).

$\Rightarrow$ $(\Delta) + \oint_C xy\,dx = 0$ (or $\underbrace{(\Delta) + \oint_C -\frac{x^2}{2}\,dy = 0}_{(*)}$).

$\Rightarrow$ $(\Delta) = 0 - \oint_C xy\,dx$

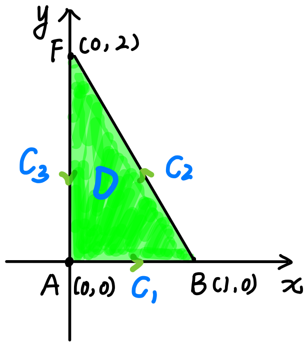
$$= 0 - \left(\int_{C_1} xy\,dx + \int_{C_2} xy\,dx + \int_{C_3} xy\,dx \right)$$

$$= 0 - \left(\int_0^1 x\cdot 0\,dx + \int_1^0 x\cdot(2-2x)\,dx + 0 \right)$$

$$= \int_0^1 x\cdot(2-2x)\,dx$$

$$= x^2 - \frac{2}{3}x^3 \Big|_{x=0}^{x=1}$$

$$= \frac{1}{3}$$



Remark:

(1). In Method 2, you can also equivalently select (☆) to do the final calculation:

$$(☆) = \frac{1}{2} \oint_C x^2\,dy$$

$$= \frac{1}{2} \left(\int_{C_1} x^2\,dy + \int_{C_2} x^2\,dy + \int_{C_3} x^2\,dy \right)$$

$$= \frac{1}{2} \left(0 + \int_0^2 \left(1 - \frac{y}{2}\right)^2 dy + \int_2^0 0^2\,dy \right)$$

$$= \frac{1}{2} \int_0^2 \left(1 - y + \frac{y^2}{4}\right) dy$$

$$= \frac{1}{2} \cdot \left(y - \frac{y^2}{2} + \frac{y^3}{12} \right) \Big|_{y=0}^{y=2}$$

$$= \frac{1}{2} \times \frac{2}{3}$$

$$= \frac{1}{3}$$

(2). For this question, method 1 and method 2 both don't involve complicated calculation, So they are both fine.

But generally speaking, it should be simpler to apply Green's Theorem directly to tackle the line integral along the closed curve.