Question

For a one-variable function, if there are two local maximum points, there is definitely a minimum point in between them. Is this true for multivariable functions as well? Let's find out!

Locate the critical points of the function $f(x, y) = 4x^2e^y - 2x^4 - e^{4y}$ and determine the nature of these points.

Solution:

$$f_x = 8x \cdot e^y - 8x^3 = 8x \cdot e^y - 8x \cdot x^2 = 8x(e^y - x^2)$$

$$f_y = 4x^2 \cdot e^y - 4 \cdot e^{4y} = 4e^y \cdot x^2 - 4e^y \cdot e^{3y} = 4e^y(x^2 - e^{3y})$$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow (*) \begin{cases} 8x \cdot (e^y - x^2) = 0 & ① \\ 4e^y(x^2 - e^{3y}) = 0 & ② \end{cases}, \quad \begin{cases} x = 0 \quad \text{or} \quad x^2 = e^y \\ x^2 = e^{3y} \end{cases}$$

For $\begin{cases} x = 0 \\ x^2 = e^{3y} \end{cases}$, $e^{3y} = 0^2 = 0$, contradiction, so rejected.

For $\begin{cases} x^2 = e^y & ③ \\ x^2 = e^{3y} & ④ \end{cases}$, $e^y = e^{3y} \Rightarrow e^y(1 - e^{2y}) = 0$

$$\Rightarrow 1 - e^{2y} = 0$$

$$\Rightarrow y = 0$$

Sub $y = 0$ into ③, $x^2 = e^0 = 1 \Rightarrow x = \pm 1$.

For $(*)$ system, it has $\begin{cases} x = 1 \\ y = 0 \end{cases}$ or $\begin{cases} x = -1 \\ y = 0 \end{cases}$

Hence, $f(x, y)$ has two critical points $(1, 0)$ and $(-1, 0)$

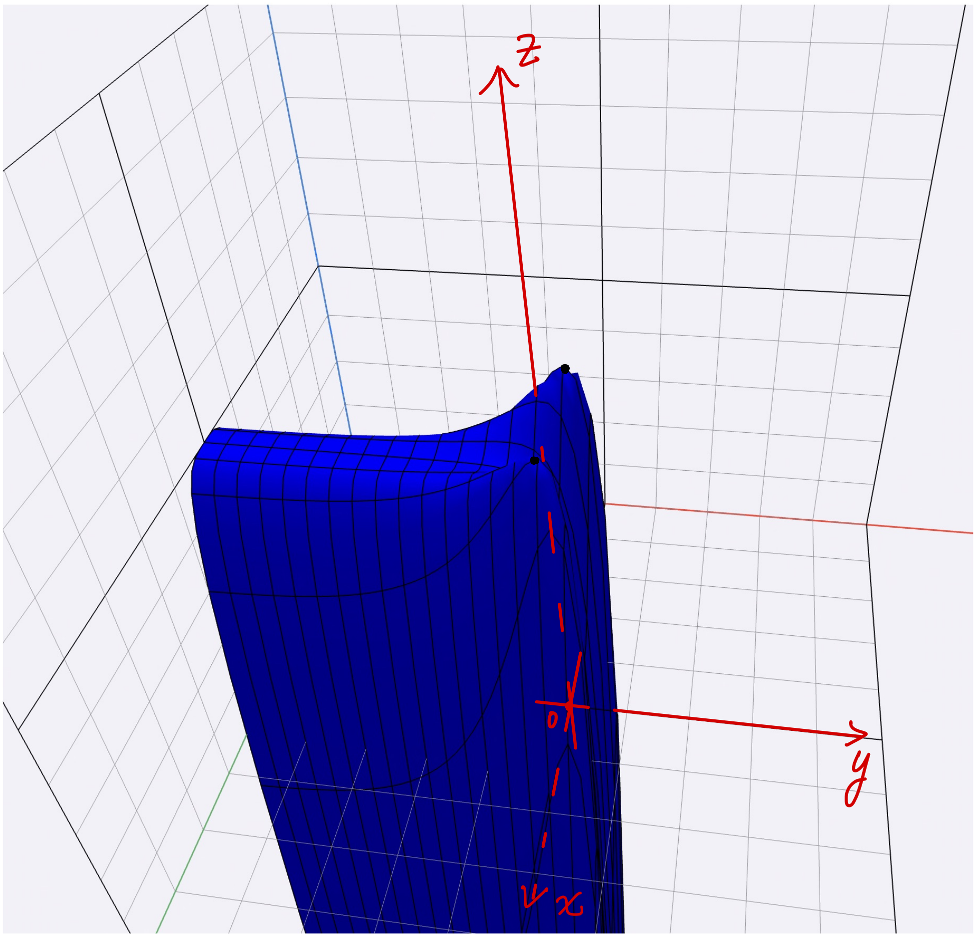
$$f_{xx} = 8 \cdot e^y - 24x^2, \quad f_{yy} = 4x^2 \cdot e^y - 16 \cdot e^{4y}, \quad f_{xy} = 8x \cdot e^y$$

$$D = f_{xx}(a,b) \cdot f_{yy}(a,b) - f_{xy}^2(a,b)$$

	f_{xx}	f_{yy}	f_{xy}	D	type
$(1, 0)$	$-16 < 0$	-12	8	$128 > 0$	local maximum
$(-1, 0)$	$-16 < 0$	-12	-8	$128 > 0$	local maximum

Hence, $f(x, y)$ has two local maximum points and there is no local minimum point between them.

Lastly, let's sketch the graph of $f(x, y)$ on the next page.


z
0
y
x