Problem:

We assume SDE $y' = Ay$, where $A_{2\times 2}$ is a real matrix, and A has only one repeated eigenvalue λ with only one linearly independent eigenvector v and a generalized eigenvector u satisfying $(\lambda I - A)u = v$.

Show that $y = (-tv + u)\cdot e^{\lambda t}$ is a solution of $y' = Ay$.

Proof:

First, on the left,

$$y' = (-1\cdot v + 0)\cdot e^{\lambda t} + (-tv + u)\cdot \lambda \cdot e^{\lambda t}$$
$$= (-v - \lambda v\cdot t + \lambda u)\, e^{\lambda t}.$$

Since v is an eigenvector of A, $Av = \lambda v$.

Thus, $y' = (-Av\cdot t - v + \lambda u)\cdot e^{\lambda t}$

On the right,

$$Ay = A(-tv + u)\, e^{\lambda t}$$
$$= (-Av\cdot t + Au)\cdot e^{\lambda t}$$

Since u is a generalized eigenvector of A,

$$(\lambda I - A)u = v$$
$$\Rightarrow \lambda u - Au = v$$
$$\Rightarrow Au = \lambda u - v = -v + \lambda u$$

Thus, $Ay = (-Avt - v + \lambda u)\cdot e^{\lambda t} = y'$.

Hence,

$y = (-tv + u)\cdot e^{\lambda t}$ is a solution of $y' = Ay$.