Question:

An engineer has found two solutions to a homogeneous system $Ax=0$ of 40 equations and 42 variables. These two solutions are not scalar multiple of each other, and every solution of the system can be constructed by a linear combination of these two solutions. Suppose now the engineer needs to replace the homogeneous system with an associated non-homogeneous system $Ax=b$ for some b, $b\neq 0$. Can the engineer be certain that he will be able to find a solution?

Solution:

We assume that u_1, u_2 are two solutions that the engineer found for homogeneous system $Ax=0$.

Since $u_1 \neq c\cdot u_2$ for all $c\in\mathbb{R}$,

then u_1 and u_2 are linearly independent.

Since every solution of $Ax=0$ can be expressed as a linear combination of u_1, u_2,

then the solution space $= \text{span}\{u_1, u_2\}$

Hence, $\{u_1, u_2\}$ is a basis for the solution space, which is also a basis for the nullspace of A.

Therefore, $\text{nullity}(A) = 2$.

Since $Ax=0$ has 40 equations and 42 variables,

then A is a 40×42 matrix.

By the Dimension theorem,

$$\begin{aligned}\text{rank}(A) &= \text{total number of columns of } A - \text{nullity}(A)\\ &= 42-2\\ &= 40 = \text{the number of leading entries in } RREF(A).\end{aligned}$$

(That is to say, in the RREF of $(A|b)$, the very last column must be non-pivot. Which is, the system $Ax=b$ is always consistent, no matter what the value of the vector b is).

e.g. One possible RREF of the corresponding augmented matrix of $Ax=b$ can be:

$$\begin{array}{c} \begin{array}{cccccc} & & & \text{40th column} & & \end{array} \\ \left(\begin{array}{ccccccc|c} 1 & 0 & \cdots & 0 & a_{1,41} & a_{1,42} & & b_1 \\ 0 & 1 & \cdots & 0 & a_{2,41} & a_{2,42} & & b_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & a_{40,41} & a_{40,42} & & b_{40} \end{array}\right)_{40\times 43} \\ \text{40th row} \end{array}$$

So for the associated non-homogeneous system $Ax=b$, it will have a solution regardless of the vector b.

Conclusion:

For the matrix $A_{m\times n}$, $m\le n$,
if A is a full rank matrix, the system $Ax=b$ $(b\ne 0)$ is always consistent. To be specific,

① when $m=n$, $Ax=b$ has only one solution $x=A^{-1}b$.

② when $m<n$, $Ax=b$ has infinitely many solutions.