Question:

$v_1 = (2, -2, 0)$, $v_2 = (6, 1, 4)$, $v_3 = (2, 0, -4)$ are three vectors in $\mathbb{R}^3$ that have their initial points at the origin.

Determine whether the three vectors lie on the same plane.

Solution:

Way 1: Assume the three vectors lie in a plane $ax + by + cz - d = 0$

Since the plane goes through the origin $(0,0,0)$,

then $a \cdot 0 + b \cdot 0 + c \cdot 0 - d = 0 \Rightarrow d = 0 \Rightarrow ax + by + cz = 0$

Then
$$\underbrace{\begin{pmatrix} 2 & -2 & 0 \\ 6 & 1 & 4 \\ 2 & 0 & -4 \end{pmatrix}}_{A} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{B}$$

Using Matlab to get

$$RREF(A|B) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The system has only the trivial solution $a = b = c = 0$.

Hence, v_1, v_2, v_3 do NOT lie on the same plane.

Way 2: We observe that v_1, v_2 are not scalar multiple of each other,

so span$\{v_1, v_2\}$ = a plane P.

v_3 is on the plane P if and only if v_3 is a linear combination of v_1, v_2.

Assume $c_1 v_1 + c_2 v_2 = v_3$, $(c_1, c_2 \in \mathbb{R})$.

Then
$$\underbrace{\begin{pmatrix} 2 & 6 \\ -2 & 1 \\ 0 & 4 \end{pmatrix}}_{A} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}}_{B}.$$

Using Matlab to get

$$RREF(A|B) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

The system has no solution.

Hence, v_1, v_2, v_3 do NOT lie on the same plane.

Way 3: v_1, v_2, v_3 are three vectors on the same plane if and only if $\text{rank}(A) \leq 2$, where A is formed by stacking v_1, v_2, v_3 horizontally or vertically.

We assume,

$$A = \begin{pmatrix} 2 & -2 & 0 \\ 6 & 1 & 4 \\ 2 & 0 & -4 \end{pmatrix}$$

Using Matlab to get,

$$\text{RREF}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, $\text{rank}(A) = 3$.

Hence, v_1, v_2, v_3 do NOT lie on the same plane.

Way 4: v_1, v_2, v_3 are three vectors on the same plane if and only if v_1, v_2, v_3 are linearly dependent.

We assume $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$.

Then

$$\underbrace{\begin{pmatrix} 2 & 6 & 2 \\ -2 & 1 & 0 \\ 0 & 4 & -4 \end{pmatrix}}_{A} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{B}$$

Using Matlab to get

$$\text{RREF}(A|B) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The system has only the trivial solution $c_1 = c_2 = c_3 = 0$

Hence, v_1, v_2, v_3 do NOT lie on the same plane.