Question:

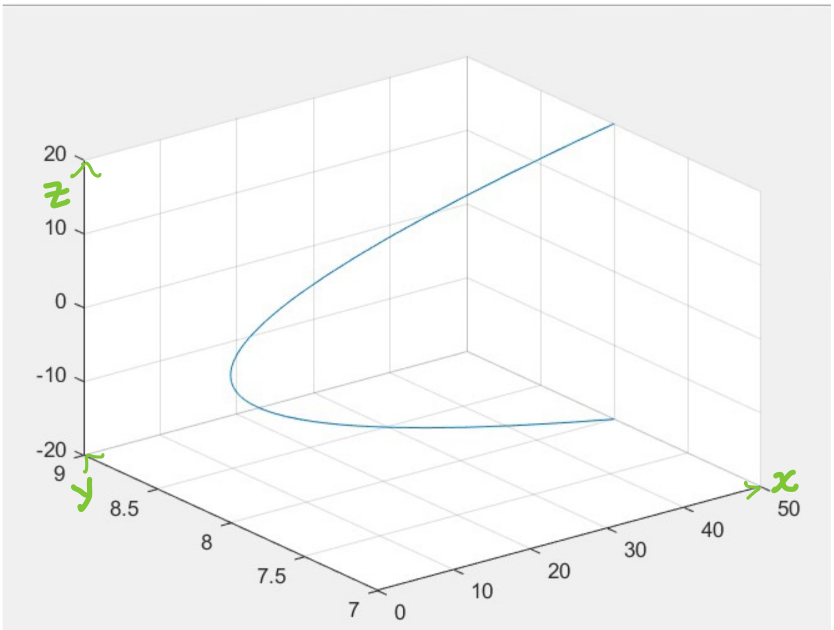
A force $F = 2x^2 i + 8j + 4xk$ moves an object along a straight line from $A(3,1,0)$ to $B(0,2,2)$.

Find the value of x that maximises the work done.

Discussion:

$$\vec{F} = \begin{pmatrix} 2x^2 \\ 8 \\ 4x \end{pmatrix},$$ where x is a parameter, and $x \in \mathbb{R}$.

So the corresponding graph of $\vec{F}$ is a curve, as shown below:



At each point of $\vec{AB}$, the force $\vec{F}$ is like the above, and its direction and magnitute both depend on x or change w.r.t. x.

That is to say, at each point of $\vec{AB}$, the angle θ between $\vec{F}$ and $\vec{AB}$, and $|\vec{F}|$ should be both related to x or some function of x respectively.

Recall:

the work done by $\vec{F}$ in this problem.

$$W = \vec{F} \cdot \vec{AB} = |\vec{F}| \cdot \underbrace{|\vec{AB}|}_{\text{constant}} \cdot \cos\theta$$

$\boxed{\text{If } |\vec{F}| \equiv \text{constant}}$, then W is a function of θ, and then

W is maximized $\Longleftrightarrow$ $\theta = 0°$ $\Longleftrightarrow$ $\vec{F}$ and $\vec{AB}$ are in the same direction.

$\Updownarrow$

$\vec{F} = \lambda \cdot \vec{AB}$ $(\lambda > 0)$

idea 1:

According to the deduction in blue, the condition for the result

W is maximized $\Longleftrightarrow$ $\vec{F} = \lambda \cdot \vec{AB}$ $(\lambda > 0)$ (*)

is $|\vec{F}| \equiv$ constant.

However, $|\vec{F}|$ in the given problem is a function of x,

and hence, the idea (*) is **NOT** applicable in this problem.

idea 2:

$$\vec{F} = \underbrace{|\vec{F}|}_{\text{a function of } x} \cdot |\vec{AB}| \cdot \underbrace{\cos\theta}_{\text{related to } x} \quad (**)$$

related to x, but the expression is unknown.

And hence, the (**) formula does **NOT** work in this problem!

correct solution:

$$F = \begin{pmatrix} 2x^2 \\ 8 \\ 4x \end{pmatrix}, \quad \vec{AB} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

$$W = F \cdot \overrightarrow{AB} = \begin{pmatrix} 2x^2 \\ 8 \\ 4x \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -6x^2 + 8 + 8x = -6x^2 + 8x + 8$$

Method 1 : complete squares.

$$\begin{aligned} W &= -6(x^2 - \tfrac{8}{6}x) + 8 \\ &= -6(x^2 - 2 \cdot x \cdot \tfrac{2}{3} + (\tfrac{2}{3})^2 - (\tfrac{2}{3})^2) + 8 \\ &= -6(x - \tfrac{2}{3})^2 + 6 \times (\tfrac{2}{3})^2 + 8 \\ &= -6(x - \tfrac{2}{3})^2 + \tfrac{32}{3} \end{aligned}$$

Hence, W is maximized at $x = \frac{2}{3}$.

Method 2 : the second derivative test

$W' = -6 \cdot 2x + 8 = -12x + 8$, let $W' = 0$, then $x = \frac{2}{3}$.

at $x = \frac{2}{3}$, $W'' = -12 \times 1 = -12 < 0$.

Hence, by the second derivative test, W is maximized at $x = \frac{2}{3}$.