Preliminary :

Theorem :

Let D be a simply connected closed region, $F(x,y) = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$.

If the functions $P(x,y)$, $Q(x,y)$ are continuous on D, with the continuous first-order partial derivatives, then the following conditions are equivalent :

(1) along any piecewise smooth closed curve C in D, $\oint_C F \cdot dr = 0$

(2) along any piecewise smooth curve C in D, $\int_C F \cdot dr$ is path independent, and it only depends on the initial point and the end point.

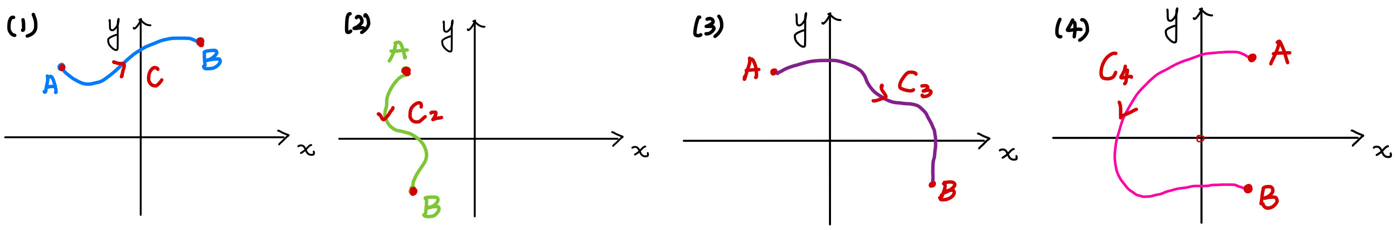
(3) On the region D, $P_y = Q_x$.

problem :

Here we will use a typical example to discuss how to obtain the suitable potential functions when applying the Fundamental Theorem of Line Integral.

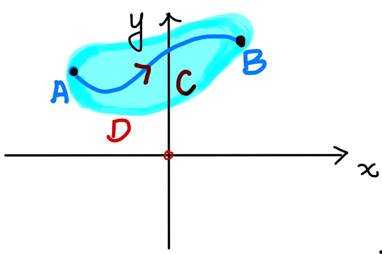
If we assume a vector field $F = \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{pmatrix}$, where F is conservative at $(x,y) \neq (0,0)$,

find a potential function f and compute $\int_C F \cdot dr$, where the curve C is given by :



Discussion :

(1)



If we sketch a simply connected region D to include the given curve C as shown,

F is conservative on the region D.

Then by definition, $F = \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{pmatrix} = \nabla f = \begin{pmatrix} f_x \\ f_y \end{pmatrix}$.

(annotations: $\frac{-y}{x^2+y^2} \to P$, $\frac{x}{x^2+y^2} \to Q$, f_y ①)

Thus,
$$f = \int \frac{-y}{x^2+y^2}\,dx = -y\cdot\int \frac{1}{x^2+y^2}\,dx$$
$$= -y\int \frac{1}{y^2\left(1+\left(\frac{x}{y}\right)^2\right)}\,dx$$
$$= -\frac{1}{y}\int \frac{1}{1+\left(\frac{x}{y}\right)^2}\,dx$$
$$= -\int \frac{1}{1+\left(\frac{x}{y}\right)^2}\,d\left(\frac{x}{y}\right)$$
$$= -\tan^{-1}\left(\frac{x}{y}\right) + g(y)$$

Differentiate the above f w.r.t. y, $f_y = -\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot x\cdot(-y^{-2}) + g'(y)$

$$\Rightarrow f_y = \frac{x}{x^2+y^2} + g'(y) \quad ②$$

Compare ① with ②, $g'(y)=0 \Rightarrow g(y) = E$

$$\Rightarrow f = -\tan^{-1}\left(\frac{x}{y}\right) + E$$

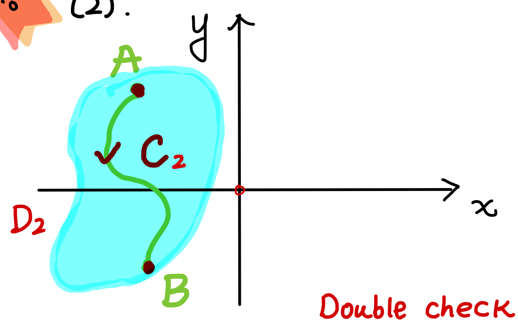
Take $E=0$ to get a potential function,

$$f = -\tan^{-1}\left(\frac{x}{y}\right).$$

✱ (Double check the above f is undefined on the x-axis, so f is well defined on D).

Hence, $\int_C F\cdot dr = f(B) - f(A) = \left(-\tan^{-1}\left(\frac{x_B}{y_B}\right)\right) - \left(-\tan^{-1}\left(\frac{x_A}{y_A}\right)\right).$

(2).



Similarly, if we sketch a simply connected region D_2 to include the given curve C_2 as shown,

F is conservative on the region D_2.

Double check the above $f = -\tan^{-1}\left(\frac{x}{y}\right)$ is undefined on the x-axis,

so f is NOT well defined on the region D_2.

Hence, $f = -\tan^{-1}\left(\frac{x}{y}\right)$ is not a potential function of F on the region D_2.

Recall

$\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2}$, $x>0$

$\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right) = -\frac{\pi}{2}$, $x<0$

$\Rightarrow$

$\underbrace{\tan^{-1}\left(\frac{y}{x}\right)}_{f_2} = \underbrace{-\tan^{-1}\left(\frac{x}{y}\right)}_{f} + \frac{\pi}{2}$, $\frac{x}{y}>0$

$\underbrace{\tan^{-1}\left(\frac{y}{x}\right)}_{f_2} = \underbrace{-\tan^{-1}\left(\frac{x}{y}\right)}_{f} - \frac{\pi}{2}$, $\frac{x}{y}<0$

$\Rightarrow$ On the common definition domain, $f_2 = f + C$, where C is a constant.

$\Rightarrow$ The difference between $f_2 = \tan^{-1}(\frac{y}{x})$ and $f = -\tan^{-1}(\frac{x}{y})$ is a constant on the common definition domain.

$\Rightarrow$ $\nabla f_2 = \nabla f = F$ $\Rightarrow$ f_2 is a potential function of F.

Double check that,

$f_2 = \tan^{-1}(\frac{y}{x})$ is only undefined on the y-axis, so f_2 is well defined on D_2.

Thus, f_2 is a potential function of F on the region D_2

Hence, $\int_{C_2} F \cdot dr = f_2(B) - f_2(A) = \tan^{-1}(\frac{y_B}{x_B}) - \tan^{-1}(\frac{y_A}{x_A})$

Remark: for the given $F = \begin{pmatrix} P \\ Q \end{pmatrix}$, you may also integrate Q w.r.t. y first, that will directly lead to the above f_2 as a potential function of F.

Continuing on the next page!