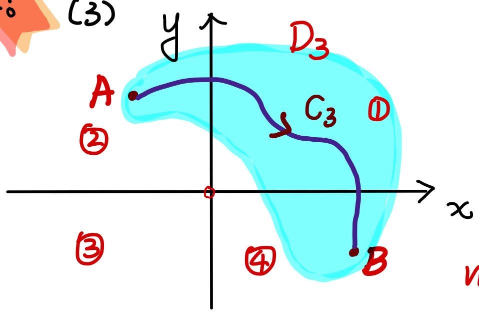
Note : this is a continuing discussion about choices of potential functions after the previous case (1) and case (2)

(3)



Similarly, if we sketch a simply connected region D_3 to include the given curve C_3 as shown

F is conservative on the region D_3.

Where ①, ②, ③, ④ are representing different quadrants.

Recall the above two potential functions,

$f = -\tan^{-1}\left(\frac{x}{y}\right)$ is only undefined at x-axis,

$f_2 = \tan^{-1}\left(\frac{y}{x}\right)$ is only undefined at y-axis,

So we can patch the two functions to attain a piecewise potential function f_3 of F on $D3$.

The rough idea is f is well defined on the region above x-axis,

f_2 is well defined on the region on the right of y-axis,

and thus the common part is the 1st quadrant ①, where

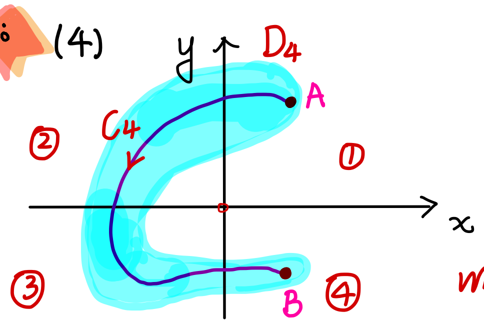
$$\tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{2}, \quad \frac{x}{y} > 0$$

$$\Rightarrow -\tan^{-1}\left(\frac{x}{y}\right) = \tan^{-1}\left(\frac{y}{x}\right) - \frac{\pi}{2}$$

Now, we can patch f and f_2 to get

$$f_3 = \begin{cases} -\tan^{-1}\left(\frac{x}{y}\right), & ① \cup ② \cup y^{+}\text{-axis}. \\ \tan^{-1}\left(\frac{y}{x}\right) - \frac{\pi}{2}, & ① \cup ④ \cup x^{+}\text{-axis} \end{cases}$$

(4)



Similarly, if we sketch a simply connected region D_4 to include the given curve C_4 as shown

F is conservative on the region D_4.

Where ①, ②, ③, ④ are representing different quadrant.

Based on the discussion in (3),

$$f_4 = \begin{cases} -\tan^{-1}\left(\frac{x}{y}\right), & ① \cup ② \cup y^+\text{-axis} \quad \text{part 1} \\ \tan^{-1}\left(\frac{y}{x}\right) + \frac{\pi}{2}, & ② \cup ③ \cup x^-\text{-axis} \quad \text{part 2} \\ -\tan^{-1}\left(\frac{x}{y}\right) + \pi, & ③ \cup ④ \cup y^-\text{-axis} \quad \text{part 3.} \end{cases}$$

Notice that for part 1 and part 2, the idea is pretty much the same as the previous case (3), where $f = -\tan^{-1}\left(\frac{x}{y}\right)$, $f_2 = \tan^{-1}\left(\frac{y}{x}\right)$.

The rough idea is f is well defined on the region above x-axis,

f_2 is well defined on the region on the left of y-axis,

and thus the common part is the 2nd quadrant ②, where

$$\tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right) = -\frac{\pi}{2}, \quad \frac{x}{y} < 0.$$

$$\Rightarrow -\tan^{-1}\left(\frac{x}{y}\right) = \tan^{-1}\left(\frac{y}{x}\right) + \frac{\pi}{2} \Rightarrow \text{the expression in part 2.}$$

For part 3,

$\underbrace{\tan^{-1}\left(\frac{y}{x}\right) + \frac{\pi}{2}}_{(**)}$ is well defined on ② ∪ ③ ∪ x^--axis,

$f = -\tan^{-1}\left(\frac{x}{y}\right)$ is well defined on the region below x-axis,

and thus the common part is the 3rd quadrant ③, where

$$\tan^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{2}, \quad \frac{x}{y} > 0$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = -\tan^{-1}\left(\frac{x}{y}\right) + \frac{\pi}{2} \quad (*)$$

Substitute (*) into (**),

$$-\tan^{-1}\left(\frac{x}{y}\right) + \frac{\pi}{2} + \frac{\pi}{2} = -\tan^{-1}\left(\frac{x}{y}\right) + \pi$$

$\Rightarrow$ the expression in part 3. ∎