Suppose the characteristic polynomial of a matrix $A_{n\times n}$ is factorized as follows, with all the common factors group together:

$$\det(\lambda I - A) = (\lambda-\lambda_1)^{r_1}\cdot(\lambda-\lambda_2)^{r_2}\cdots(\lambda-\lambda_k)^{r_k}.$$

Where $\lambda_1, \lambda_2 \dots \lambda_k$ are all eigenvalues of $A_{n\times n}$, r_1 to r_k are the respective multiplicities of these eigenvalues, with $r_1 + r_2 + \cdots + r_k = n$.

Question:

Prove that $\dim(E_{\lambda_i}) \le r_i$ for all i.

Proof:

Suppose that $A \in$ Matrix $n\times n$, and λ_0 is an eigenvalue of $A_{n\times n}$, and $\{v_1, v_2, \dots, v_\ell\}$ is a basis of E_{λ_0},

and thus, $\dim(E_{\lambda_0}) = \ell$. $(*)$

Next, extend these v_j's $(j=1,2,\dots,\ell)$ to a basis of $\mathbb{R}^n$, which can be denoted by $\{v_1, v_2, \dots, v_\ell, v_{\ell+1}, \dots, v_n\}$.

Then, we may form a matrix $P = (v_1\ v_2 \cdots v_\ell\ v_{\ell+1} \cdots v_n)$.

(Since $v_1, v_2 \dots v_n$ are n linearly independent vectors in $\mathbb{R}^n$, then P is invertible.)

Consider $D = P^{-1}AP$, and let $w_j = P^{-1}v_j$, $j = 1, 2, \dots, \ell$.

① Claim: w_j is an eigenvector of D associated with the same eigenvalue λ_0.

$$\begin{aligned} D\cdot w_j = P^{-1}AP\cdot P^{-1}v_j &= P^{-1}Av_j \\ &= P^{-1}(Av_j) \\ &= P^{-1}\cdot\lambda v_j \\ &= \lambda\cdot(P^{-1}v_j) \\ &= \lambda\cdot w_j, \quad j = 1, 2, \dots, \ell. \end{aligned}$$

Hence $D\cdot w_j = \lambda\cdot w_j$, $j = 1, 2, \dots, \ell$

② Claim: $w_j = e_j$ for $j = 1, 2, \dots \ell$, where e_j is the standard basis vector in $\mathbb{R}^n$, i.e. $e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\rightarrow$ jth row

Recall, we have defined $P = (v_1 \ v_2 \ \cdots \ v_n)$,

and $w_j = P^{-1} v_j$ = the j_{th} column of the matrix $(\underbrace{P^{-1}v_1}_{1st\ column} \ P^{-1}v_2 \ \cdots \ \underbrace{P^{-1}v_j}_{j_{th}\ column} \ \cdots \ \underbrace{P^{-1}v_n}_{n_{th}\ column})$

= the j_{th} column of the matrix $P^{-1}\underbrace{(v_1 \ v_2 \ \cdots \ v_j \ \cdots \ v_n)}_{P}$.

= the j_{th} column of the matrix I_n (as $P^{-1}P = I_n$ from the above).

$= e_j$, $j = 1, 2, \ldots, \ell$.

From the claim ① and ②,

$D \cdot e_j = \lambda_0 \cdot e_j$ for $j = 1, 2, 3, \ldots, \ell$, and then we can write D as

$$\begin{array}{c c} & \ell\text{th column} \\ \ell\text{th row} & \left(\begin{array}{cccc|c} \lambda_0 & & & & \\ & \lambda_0 & & & D_1 \\ & & \ddots & & \\ & & & \lambda_0 & \\ \hline & & O & & D_2 \end{array}\right)_{n\times n} \end{array} = D$$

where $D_1 \in$ Matrix $\ell \times (n-\ell)$

$D_2 \in$ Matrix $(n-\ell)\times(n-\ell)$

$O \in$ Matrix $(n-\ell)\times \ell$

$$\Rightarrow \lambda I - D = \begin{array}{c c} & \ell\text{th column} \\ \ell\text{th row} & \left(\begin{array}{cccc|c} \lambda-\lambda_0 & & & & \\ & \lambda-\lambda_0 & & & -D_1 \\ & & \ddots & & \\ & & & \lambda-\lambda_0 & \\ \hline & & O & & \lambda I_{n-\ell} - D_2 \end{array}\right)_{n\times n} \end{array}$$

So, the characteristic polynomial of D is

$$\det(\lambda I - D) = (\lambda - \lambda_0)^{\ell} \cdot \det(\lambda I_{n-\ell} - D_2)$$

③ claim: $\det(\lambda I - A) = \det(\lambda I - D)$

$\det(\lambda I - D) = \det(\lambda I - P^{-1}AP)$ $\longrightarrow$ $\lambda I = \lambda P^{-1}P = P^{-1}\cdot(\lambda I)\cdot P$

$= \det(P^{-1}(\lambda I - A)\cdot P)$ $\qquad \Rightarrow \lambda I - P^{-1}AP = P^{-1}(\lambda I - A)\cdot P$

$= \det(P^{-1}) \cdot \det(\lambda I - A) \cdot \det(P)$, where P is non-singular.

$= \dfrac{1}{\det(P)} \cdot \det(\lambda I - A) \cdot \det(P)$

$= \det(\lambda I - A)$.

Hence, $\det(\lambda I - A) = (\lambda - \lambda_0)^{\ell} \cdot \det(\lambda I_{n-\ell} - D_2)$

So, we can conclude that

the multiplicity of $\lambda_0 \geq \ell$, i.e. $r_0 \geq \ell$.