Question:

How can we prove

0 is an eigenvalue of $A \iff A$ is singular (*)

0 is NOT an eigenvalue of $A \iff A$ is non-singular.

Proof:

Recall the definition: $\det(\lambda I - A) = 0 \iff \lambda$ is an eigenvalue of A.

① "$\Rightarrow$" of (*).

If 0 is an eigenvalue of A, $\det(0 \cdot I - A) = 0$

$$\Rightarrow \det(-A) = 0$$

$$\Rightarrow \det(A) = 0$$

$\Rightarrow A$ is singular.

② "$\Leftarrow$" of (*).

If A is singular, $\det(A) = 0$

$$\Rightarrow \det(-A) = 0$$

$$\Rightarrow \det(0 \cdot I - A) = 0$$

$\Rightarrow 0$ is an eigenvalue of A.

putting ① and ② together to get,

0 is an eigenvalue of $A \iff A$ is singular (*)

Since we have proved (*) is true, the contraposition of (*) should be also true, which is,

0 is NOT an eigenvalue of $A \iff A$ is non-singular.