Question :

If $Y' = AY$ is a first-order homogeneous linear SDE. $A_{2\times2}$ has repeated eigenvalue λ_0 with only a single linearly independent eigenvector v.

Then a generalized eigenvector u can be found from $(A-\lambda I)u = v$. (*)

Show that there are infinitely many solutions for u.

Proof :

As v is an eigenvector of $A_{2\times2}$, then

$$(A-\lambda_0 \cdot I)\cdot v = 0 \quad ①$$

For any non-zero vector $w \in \{ \mathbb{R}^2 \setminus \mathbb{R}\cdot v \}$,

$$(A-\lambda_0 \cdot I)\, w \neq 0 .$$

Since λ_0 is the repeated eigenvalue of $A_{2\times2}$, then the characteristic polynomial of A is

$$f(\lambda) = (\lambda - \lambda_0)^2 .$$

According to Hamilton – Cayley theorem,

$$f(A) = (A - \lambda_0 I)^2 = O_{2\times2} .$$

And thus, $(A-\lambda_0 I)^2 \cdot w = O_{2\times2}\cdot w = O_{2\times1}$,

$$\Rightarrow (A-\lambda_0 I)\cdot (A-\lambda_0 I)w = 0 \quad ②$$

Compare ① with ②, $(A-\lambda_0 I)w$ is an eigenvector of A.

$\Rightarrow (A-\lambda_0 I)\cdot w = C_w \cdot v$, where C_w is a non-zero constant associated with the vector w.

$\Rightarrow (A-\lambda_0 I)\cdot \frac{1}{C_w} w = v$

$\Rightarrow \frac{1}{C_w} w = u$ is a solution of the system $(A-\lambda_0 I)\cdot u = v$. (*)

Recall w is any non-zero vector $\in \{ \mathbb{R}^2 \setminus \mathbb{R}\cdot v \}$,

$\Rightarrow$ there are infinitely many solutions for u.