Question :

Find the exact value of the line integral

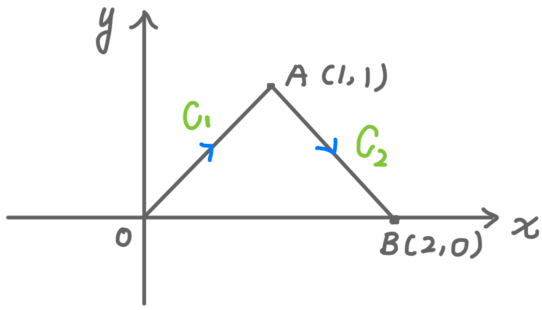
$$\int_C (x^2+y^2)\,dx + (x^2-y^2)\,dy,$$

Where C consists of the line segment from $(0,0)$ to $(1,1)$, followed by the line segment from $(1,1)$ to $(2,0)$.

Solution :

Method 1. doing it in the vector form.

Recall : the line integral of vector field, $\int_C F\cdot dr = \int_a^b F(r(t))\cdot r'(t)\,dt$



$$\int_C (x^2+y^2)\,dx + (x^2-y^2)\,dy = \int_C F\cdot dr, \quad \text{with } F=\begin{pmatrix} x^2+y^2 \\ x^2-y^2 \end{pmatrix},\ dr=\begin{pmatrix} dx \\ dy \end{pmatrix}.$$

$$= \int_{C_1} F\cdot dr + \int_{C_2} F\cdot dr \quad (*)$$

For C_1, $r_1(t) = (1-t)\cdot\begin{pmatrix} 0 \\ 0 \end{pmatrix} + t\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix}$, $0\le t\le 1$.

AS $F(r_1(t)) = \begin{pmatrix} t^2+t^2 \\ t^2-t^2 \end{pmatrix} = \begin{pmatrix} 2t^2 \\ 0 \end{pmatrix}$, $r'(t)=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$,

then $\int_{C_1} F\cdot dr = \int_0^1 \begin{pmatrix} 2t^2 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \int_0^1 2t^2\,dt = \frac{2}{3}$

For C_2, $r_2(t) = (1-t)\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} + t\cdot\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

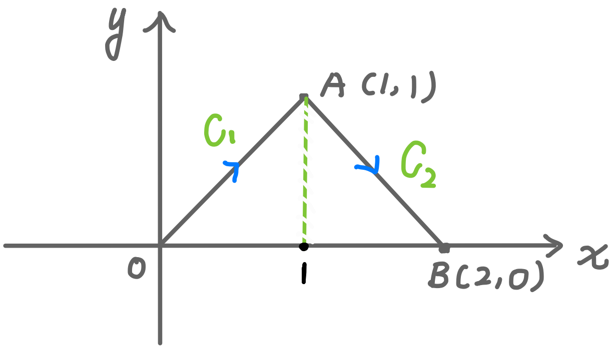
$$= \begin{pmatrix} 1-t \\ 1-t \end{pmatrix} + \begin{pmatrix} 2t \\ 0 \end{pmatrix} = \begin{pmatrix} 1+t \\ 1-t \end{pmatrix}, \quad 0\le t\le 1.$$

As $F(\gamma_2(t)) = \begin{pmatrix} (1+t)^2 + (1-t)^2 \\ (1+t)^2 - (1-t)^2 \end{pmatrix} = \begin{pmatrix} 2+2t^2 \\ 4t \end{pmatrix}$, $\gamma'(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

then $\int_{C_2} F \cdot dr = \int_0^1 \begin{pmatrix} 2+2t^2 \\ 4t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} dt = \int_0^1 (2t^2 - 4t + 2)\, dt = \underline{\frac{2}{3}}$

Hence, $(*) = \frac{2}{3} + \frac{2}{3} = \underline{\frac{4}{3}}$.

Method 2. doing it in component form directly.



C_1: $y = x$, $0 \le x \le 1$.

C_2: $y = -x+2$, $1 \le x \le 2$

$$\int_C (x^2+y^2)\,dx + (x^2-y^2)\,dy$$

$$= \int_{C_1} (x^2+y^2)\,dx + (x^2-y^2)\,dy + \int_{C_2} (x^2+y^2)\,dx + (x^2-y^2)\,dy$$

$$= \int_0^1 (x^2+x^2)\,dx + (x^2-x^2)\,dx + \int_1^2 (x^2+(-x+2)^2)\,dx + (x^2-(-x+2)^2)\,d(-x+2)$$

$$= \int_0^1 2x^2\,dx + \int_1^2 \Big((x^2+(-x+2)^2) - (x^2-(-x+2)^2) \Big)\,dx$$

$$= \int_0^1 2x^2\,dx + \underbrace{\int_1^2 2(2-x)^2\,dx}_{(*)}$$

$$= \frac{2}{3} + \frac{2}{3}$$

$$= \underline{\frac{4}{3}}.$$

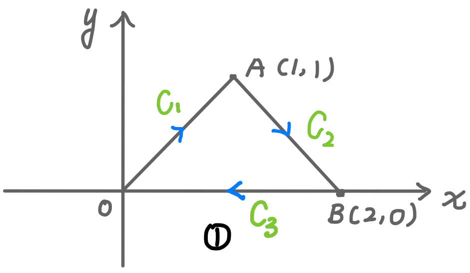
$$(*) = 2\int_1^2 (x^2 - 4x + 4)\,dx$$

$$= 2 \cdot \left(\frac{x^3}{3} - 2x^2 + 4x\right)\Big|_{x=1}^{x=2}$$

$$= 2 \cdot \left(\left(\frac{8}{3} - 8 + 8\right) - \left(\frac{1}{3} - 2 + 4\right) \right)$$

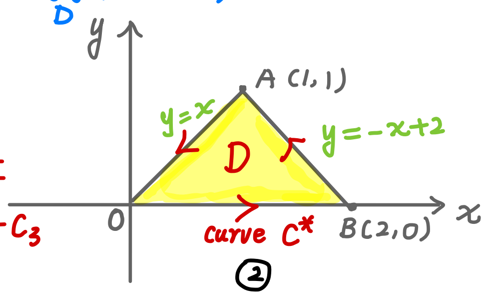
$$= \frac{2}{3}$$

Method 3. Green's Theorem. $\oint_{C^*} F\cdot dr = \iint_D (Q_x - P_y)\, dA$



$C = C_1 + C_2$

$-C^* = C_1 + C_2 + C_3$
$= C + C_3$
So, $C = -C^* - C_3$



First, we write the given component form back to the vector form,

$$\int_C (x^2+y^2)\,dx + (x^2-y^2)\,dy = \int_C F\cdot dr, \quad \text{with } F = \begin{pmatrix} x^2+y^2 \\ x^2-y^2 \end{pmatrix} \begin{matrix} \to P \\ \to Q \end{matrix}, \quad dr = \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

Then based on the diagram ① and ②,

$$C = -C^* - C_3 \Rightarrow \int_C F\cdot dr = \int_{-C^*} F\cdot dr + \int_{-C_3} F\cdot dr$$
$$= \underbrace{-\int_{C^*} F\cdot dr}_{(A)} + \underbrace{\int_{-C_3} F\cdot dr}_{(B)} \quad (**)$$

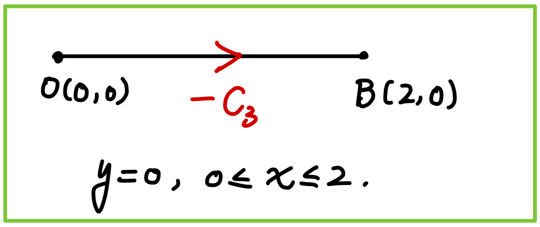
For (A), $P_y = 2y$, $Q_x = 2x$,
region $D = \{(x,y): 0 \le y \le 1,\ y \le x \le 2-y\}$,

$$(A) = -\iint_D (2x-2y)\, dA$$
$$= -\int_0^1 \underbrace{\int_y^{2-y} (2x-2y)\,dx}_{(*)}\,dy$$
$$= -\int_0^1 (4y^2 - 8y + 4)\, dy$$
$$= -\frac{4}{3}$$

$$(*) = x^2 - 2y\cdot x \Big|_{x=y}^{x=2-y}$$
$$= \left((2-y)^2 - 2y(2-y)\right) - (y^2 - 2y^2)$$
$$= 4y^2 - 8y + 4$$

For (B), we can apply method 1 or method 2.

Perform method 2,



$$(B) = \int_0^2 (x^2 + 0^2)\,dx + (x^2 - 0^2)\,d0$$
$$= \int_0^2 x^2\,dx = \frac{8}{3}$$

Hence, $(**) = -\frac{4}{3} + \frac{8}{3} = \frac{4}{3}$.