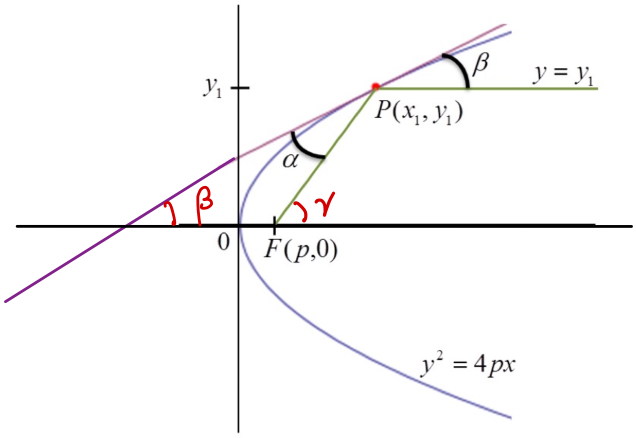
Question:

Let $P(x_1, y_1)$ be a point on the curve $y^2 = 4px$. This shape is called a parabola. The parabola and its tangent at point P are drawn on the diagram below. Let $F(p, 0)$ be a point on the $x-$axis. Let β be the (acute) angle between $y = y_1$ and the tangent of the parabola. Let α be the (acute) angle between FP and the tangent of the parabola.

Show that $\alpha = \beta$.



Solution:

Method 1.

Recall: $\tan\beta$ = the gradient of tangent line to the curve $y^2 = 4px$ at the the point $P(x_1, y_1) = y'\big|_{x=x_1, y=y_1}$

Since $2y \cdot y' = 4p \Rightarrow y' = \frac{2p}{y}$, $\Rightarrow \tan\beta = y'\big|_{x=x_1, y=y_1} = \frac{2p}{y_1}$

Since $\gamma = \alpha + \beta$, $\Rightarrow \tan\gamma = \tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta}$.

In the meantime, $\tan\gamma = \dfrac{y_P - y_F}{x_P - x_F} = \dfrac{y_1 - 0}{x_1 - p} = \dfrac{y_1}{x_1 - p}$

$$\Rightarrow \frac{y_1}{x_1 - p} = \frac{\tan\alpha + \frac{2p}{y_1}}{1 - \tan\alpha \cdot \frac{2p}{y_1}}$$

$$\Rightarrow y_1 - 2p \cdot \tan\alpha = (x_1 - p) \cdot \tan\alpha + \frac{2p}{y_1} \cdot (x_1 - p)$$

$$= (x_1 - p)\tan\alpha + \frac{2px_1}{y_1} - \frac{2p^2}{y_1}$$

$$\Rightarrow y_1 - \frac{2px_1}{y_1} + \frac{2p^2}{y_1} = (x_1 + p)\tan\alpha$$

$$\Rightarrow \tan\alpha = \frac{y_1 - \frac{2px_1}{y_1} + \frac{2p^2}{y_1}}{x_1 + p} = \frac{\overset{=4px_1}{y_1^2} - 2px_1 + 2p^2}{y_1(x_1+p)}$$

$$= \frac{2px_1 + 2p^2}{y_1(x_1+p)}$$

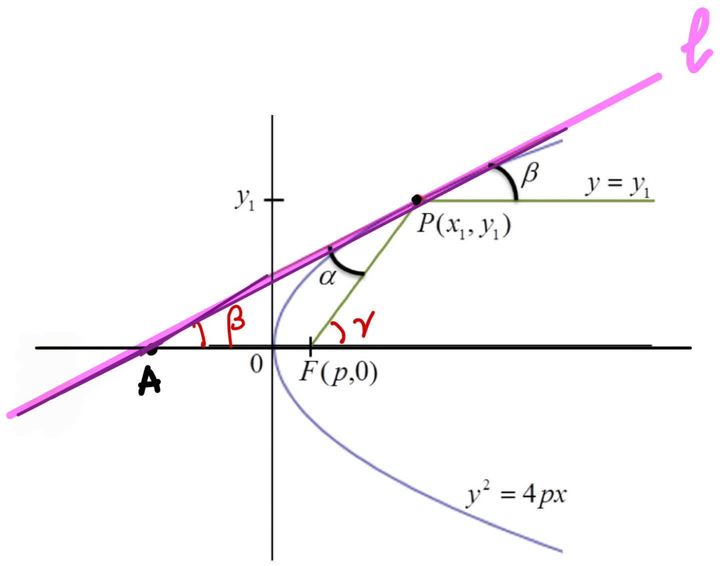
$$= \frac{2p(x_1+p)}{y_1(x_1+p)}.$$

$$= \frac{2p}{y_1} = \tan\beta.$$

Since $0 < \alpha, \beta < \frac{\pi}{2}$, then

$\boxed{\alpha = \beta}$.

Method 2.



From Method 1,

the equation of the tangent line ℓ is $y - y_1 = \frac{2p}{y_1}(x - x_1)$

$$\text{Let } y = 0, \quad 0 - y_1 = \frac{2p}{y_1}(x - x_1)$$

$$\frac{-y_1^2}{2p} = x - x_1$$

$$\Rightarrow x = \frac{-y_1^2}{2p} + x_1$$

Since $P(x_1, y_1)$ is one point on the parabola,

$$y_1^2 = 4px_1 \Rightarrow -\frac{y_1^2}{2p} = -\frac{4px_1}{2p} = -2x_1$$

$$\Rightarrow x = -2x_1 + x_1 = -x_1$$

$\Rightarrow$ the point $A(-x_1, 0)$

$$|AF| = p - (-x_1) = p + x_1$$

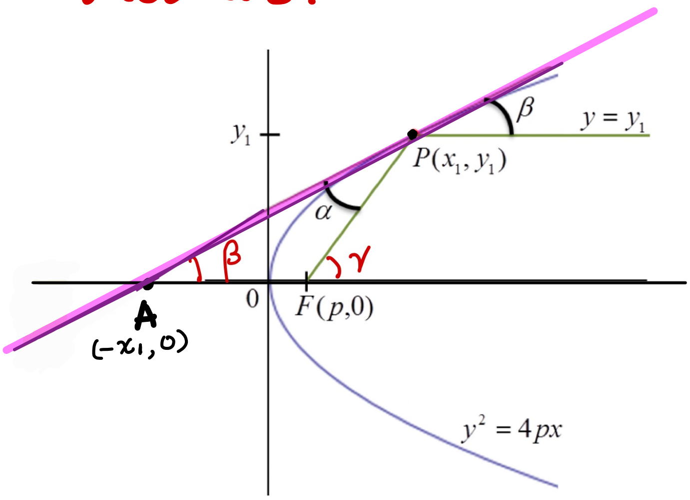
$$|FP| = \sqrt{(y_1 - 0)^2 + (x_1 - p)^2} = \sqrt{(x_1 - p)^2 + y_1^2}$$

$$\Rightarrow |FP|^2 = (x_1 - p)^2 + y_1^2 = (x_1 - p)^2 + 4px_1 = x_1^2 - 2px_1 + p^2 + 4px_1$$

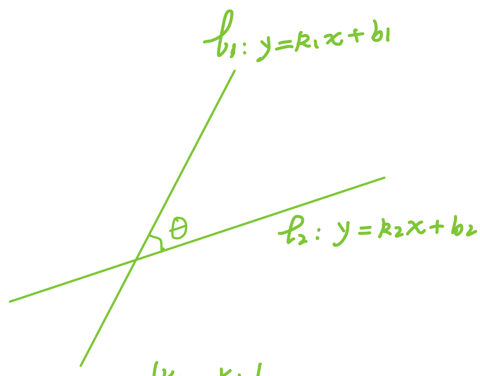
$$= (x_1 + p)^2 = |AF|^2$$

$\Rightarrow |AF| = |FP| \Rightarrow \triangle AFP$ is an isosceles triangle $\Rightarrow \boxed{\alpha = \beta}$

Method 3.



Recall:



$$\tan\theta = \frac{|k_2 - k_1|}{1 + k_1 \cdot k_2}$$

Solution:

Since the gradient / slope of tangent line $l = \tan\beta = \frac{2p}{y_1} = k_{AP}$.

$$k_{FP} = \frac{y_1 - 0}{x_1 - p} = \frac{y_1}{x_1 - p}$$

$$\tan\alpha = \frac{|k_{AP} - k_{PF}|}{1 + k_{AP} \cdot k_{PF}} = \frac{\left|\frac{2p}{y_1} - \frac{y_1}{x_1 - p}\right|}{1 + \frac{2p}{y_1} \cdot \frac{y_1}{x_1 - p}} \quad \begin{matrix} \cdot\, y_1(x_1 - p) \\ \cdot\, y_1(x_1 - p) \end{matrix}$$

$$= \frac{|2p(x_1 - p) - y_1^2|}{y_1(x_1 - p) + 2p y_1}$$

$$= \frac{|2px_1 - 2p^2 - y_1^2|}{y_1(x_1 - p + 2p)} \quad y_1^2 \rightarrow 4px_1$$

$$= \frac{|-2px_1 - 2p^2|}{y_1(x_1 + p)}$$

$$= \frac{|-(2px_1 + 2p^2)|}{y_1 \cdot (x_1 + p)} \quad (x_1 > 0,\ p > 0)$$

$$= \frac{2p(x_1 + p)}{y_1(x_1 + p)} = \frac{2p}{y_1} = \tan\beta$$

Since $0 < \alpha, \beta < \frac{\pi}{2}$, $\tan\alpha = \tan\beta$, then

$$\boxed{\alpha = \beta.}$$