Question

The definite integration

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

can be derived using the following method. Let

$$I = \int_0^\infty e^{-x^2}\,dx$$

(a) Give a reasonable argument to show that

$$\underbrace{I^2}_{\text{LHS}} = \underbrace{\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy}_{\text{RHS}}$$

(b) Convert the above double integral to one wittern in polar coordinates (r, θ).

(c) Solve the double integral to obtain I^2.

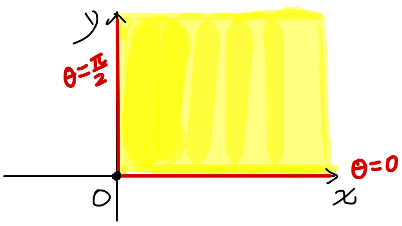
Solution:

(a) $$\text{RHS} = \int_0^\infty \int_0^\infty e^{-x^2} \cdot e^{-y^2}\,dx\,dy$$

$$= \int_0^\infty e^{-y^2} \int_0^\infty e^{-x^2}\,dx\,dy$$

$$= \int_0^\infty e^{-x^2}\,dx \cdot \int_0^\infty e^{-y^2}\,dy = I \cdot I = I^2$$

(b). The domain of integration in Cartesian coordinate is



$$D = \{(x, y) : x \geq 0,\ y \geq 0\},$$

and in polar coordinate is

$$D = \{(r, \theta) : r \geq 0,\ 0 \leq \theta \leq \tfrac{\pi}{2}\}$$

Rewrite the RHS in polar coordinate,

$$\text{RHS} = \int_0^{\frac{\pi}{2}} \underbrace{\boxed{\int_0^\infty e^{-r^2}\, r\,dr}}_{(*)}\,d\theta$$

(c).

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\,d\theta$$

$$= \frac{1}{2} \cdot \int_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{4} = \text{LHS} = I^2 \quad \Rightarrow \quad I = \frac{\sqrt{\pi}}{2} = \int_0^{+\infty} e^{-x^2}\,dx.$$

$r\,dr = -\frac{1}{2} \cdot d(-r^2)$

$$(*) = \int_0^\infty e^{-r^2} \cdot$$

$$= -\frac{1}{2} \int_0^\infty e^{-r^2}\,d(-r^2)$$

$$= -\frac{1}{2} \cdot e^{-r^2} \Big|_{r=0}^{r=\infty}$$

$$= -\frac{1}{2}(0 - 1) = \frac{1}{2}$$