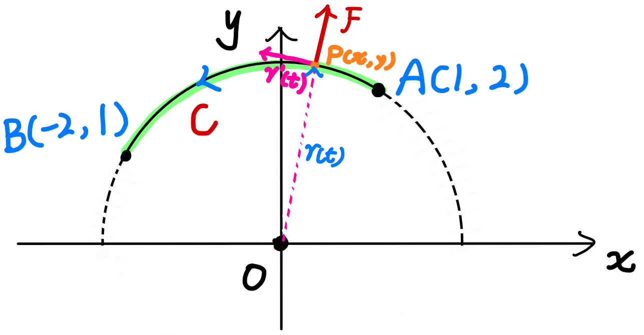
Problem :

Here we will use an example to show a special perpespective in dealing with the line integral of a vector field.

If a vector field $F = \begin{pmatrix} \frac{x}{2x^2+y^2} \\ \frac{y}{2x^2+y^2} \end{pmatrix}$, find the exact value of $\int_C F \cdot dr$,

where C is a part of the curve $x^2+y^2=5$, $y>0$, that joins from $A(1,2)$ to $B(-2,1)$.



Discussion :

(1) general idea 1 :

We can show F is conservative at $(x,y) \neq (0,0)$, and then apply the Fundamental Theorem of Line integral $\int_C F \cdot dr = f(B) - f(A)$.

(2) general idea 2 :

Apply the general formula for the line integral of a vector field, $\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\,dt$

(3) special perspective :

Let $P(x,y)$ = an arbitrary point on the given curve C.

$r(t)$ = a parametric equation of the curve C.

Then $r^2(t) = x^2(t) + y^2(t) = 5 \Rightarrow \frac{d(r^2(t))}{dt} = 0$

$\Rightarrow r(t) \cdot r'(t) = 0$

$\Rightarrow r(t) \perp r'(t)$

As $F = \begin{pmatrix} \frac{x}{2x^2+y^2} \\ \frac{y}{2x^2+y^2} \end{pmatrix} = \frac{1}{2x^2+y^2} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{OP}}$ = multiple of $\vec{OP}$

= multiple of $r(t)$,

$\Longrightarrow F \parallel r(t) \Rightarrow F \perp r'(t)$

If we assume there is a particle moving along the curve C from A to B,

then $\quad r'(t) \longrightarrow$ the velocity of this particle at the time t

$F \longrightarrow$ the force in moving this particle.

$\int_C F \cdot dr \rightarrow$ the work done by F in moving this particle.

Then, $\quad F \perp r'(t)$

$\Rightarrow$ the force $\perp$ the velocity

$\Rightarrow$ the work done by F is 0

$\Rightarrow \int_C F \cdot dr = 0$

Conclusion:

In general, if the curve C is given by a part of or the whole circle

$x^2 + y^2 = a^2,\ a > 0$, a vector field $F = \begin{pmatrix} g(x,y) \cdot x \\ g(x,y) \cdot y \end{pmatrix} = g(x,y) \cdot \begin{pmatrix} x \\ y \end{pmatrix}$,

then $\int_C F \cdot dr = 0$.